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A perishable inventory optimization model for two commodities in multi-echelon system

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Abstract

In this article, we delve into the quest for an optimal solution within the realm of a practical inventory system, which serves as an integral component of a larger Multiechelon inventory system. Our study focuses on a warehouse (WH) that houses a single Distribution Center (DC) and a sole retailer (R) tasked with managing two distinct yet interconnected perishable products. This setup is modeled as a three-stage continuous review inventory system.

At the retailer node, we implement an (s, S) inventory system, accommodating Poisson demand and exponentially distributed lead times. It's important to note that the primary products are susceptible to perishing exclusively at the retailer node, characterized by a rate ζ, whereas the sub-products are non-perishable in nature. Deliveries to retailers occur in packs of $Q_i(= S_i - s_i)$ units, originating from the distribution center (DC) with immediate replenishment capabilities sourced from the well-stocked warehouse (WH).

Throughout our analysis, we rigorously maintain steady-state probability distributions and behavioral insights. Furthermore, we diligently collect and analyze performance measurements for the system under steady-state conditions. Our findings are showcased through numerical examples, shedding light on the practical implications of our inventory management approach.

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1 Introduction

Inventory management is a critical aspect of efficient business operations. Whether dealing with a single item at a single location or a complex, multi-item, multi-location scenario, the proper control of inventory is central to a company's success. This article explores the dynamic landscape of inventory management, from its fundamental principles to its practical applications in a world where maintaining stocks of goods is commonplace.

In the business world, maintaining an inventory, or a stockpile of products, is essential to meet future demand and ensure timely deliveries to customers. The ultimate goal of inventory theory is to provide management with strategies and rules to minimize the costs associated with maintaining inventory while meeting customer demands effectively. Effective inventory management can translate into substantial cost savings for businesses. At its core, inventory theory seeks to answer two pivotal questions: When should an order be placed for a product? And how large should each order be? The amalgamation of responses to these questions constitutes an inventory policy.

In certain business scenarios, companies handle not just one product but a pair of different yet interrelated products. These product pairs, such as tires and tubes, pens and refills, printers and cartridges, or insulin vials and injections, share a common manufacturing unit and exhibit distinct demand patterns. Initially, customers may need to purchase the main product alongside the subproduct, but as they become familiar with the offerings, they may choose to acquire the subproduct separately. Typically, there is a higher demand for the subproduct compared to the main product, making it essential to manage these variations effectively.

Perishable goods, with their finite or specified lifespans, present unique challenges in inventory management. Real-world examples include items like milk, blood, drugs, food, vegetables, and certain chemicals, all of which have limited shelf lives. Handling such perishable inventory systems requires meticulous planning and management. These goods, once expired, not only occupy valuable storage space but can also affect neighboring items. Hence, it is crucial to remove items that have exceeded their shelf life at specific intervals.

Inventory selection plays a pivotal role in the realm of Multi-echelon management, where inventory exists in diverse forms, including perishable goods, semi-

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finished products, or packaged items, at different points in the supply chain. It can also be in transit between various levels or stations as a work in progress. Given that inventory holdings can account for a substantial portion, ranging from 20 % to 40 $\%$, of their overall value, effective inventory management is indispensable in Multi-echelon operations.

The various inventory systems (s, S) for individual products have been very extensively investigated in the past. The primary quantitative inventory survey began with the study of Harris [1915]. Clark and Scarf [1960] first proposed multi-level inventory. They analyzed Nesheron's pipeline framework without considering partial measures. Subsequent advances in the two-stage model can be found in He et al. [1998]. Axsater [1990] proposed an approximate model of inventory structure in SC. One of the oldest papers in the field of continuous review multi-echelon inventory system is a basic and seminal paper written by Sherbrooke [1998]. A Complete review was provided by Benita [1998]. The Multi-echelon concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic work of Clark and Scarf [1960] . A continuous review perishable inventory system at Service Facilities was studied by Elango [2001]. A continuous review (s, S) policy with positive lead times in two-echelon Multi-echelon in single item was considered by Krishnan [2007].

For an inventory system with continuous review, Krishnamoorthy et al. [1994] addressed the inventory problem of two commodities. Kalpakam and Arivarignan [1993] analyzed inventory models with multiple items and renewal requirements under a common replenishment policy. Anbazhagan and Arivarignan [2000] analyzed two inventory systems under different ordering policies. Yadavalli et al. [2004] analyzed the model using a common order policy and different order quantities. Bakthavachalam et al. [2012] addressed stochastic perishable inventory control with partial backorders. Kumaresan et al. [2014] explored perishable inventory systems with (s, Q) policies. Kartheeswari and Bakthavachalam [2017] analyzed integrated inventory control systems with partial backlogging. These works collectively advance the understanding of optimizing perishable inventory management in supply chains.

This article is structured as follows: Section 2 presents a comprehensive mathematical model of the inventory management problem, along with crucial notations used throughout our exploration. In Section 3, we delve into both transient and steady-state analyses of the inventory system. Section 4 offers insights into the operational characteristics of the system, while Section 5 focuses on the cost analysis associated with inventory operations. Finally, we conclude our exploration in Section 6.

2 The Mathematical Model

The multi-echelon system under consideration in this article can be described as follows: It comprises a serial multi-echelon system composed of a warehouse facility (WH), a distribution center (DC), and a retailer (R), handling two distinct yet interconnected products. Initially, the finished product is dispatched from WH to DC following a $(0, M)$ replenishment policy. Subsequently, the product is transported to R, and the (s_i, S_i) policy $(i = 1, 2)$ is applied. Demand at the retailer node follows independent Poisson distributions with rates λ_A, λ_B, and λ_C for the main product (A), sub-product (B), or both products.

If demand occurs during the stock period and cannot be met, it is considered a lost sale. The main product is subject to perishability at a rate ξ, but only while it is at the retailer nodes. In contrast, sub-products are assumed to be non-perishable. Supplies to the manufacturer, delivered in packets of Q items, follow an exponential time interval with a parameter $\mu(> 0)$. Replenishment of items in the form of packets, from WH to DC, occurs instantaneously. In this model, the maximum inventory level at the R node is denoted as S_i, with a corresponding reorder level of $s_i(i = 1, 2)$. The ordering quantity is defined as $Q = S_i - s_i(i = 1, 2)$, and the maximum inventory level at DC_i is $M_i(M_i = nQ_i)$.

Parameters	Particulars
$[R]_{ij}$	The element /sub matrix at (i,j)th position of R.
0	Zero matrix
I	Identity matrix.
e	A column vector of ones of appropriate dimension.
k_D	Fixed ordering cost, regardless of order size at DC nodes.
k_A	The average ordering cost for main product at retailer node.
K_B	The average ordering cost for sub product at retailer node.
h_D	The holding cost per unit of item per unit time at DC nodes.
h_A	The average holding cost per unit of item per unit time for main product at retailer.
h_B	The average holding cost per unit of item per unit time for sub product at retailer.
g_A	The average unit shortage cost for main product at retailer nodes.
g_B	The average unit shortage cost for sub product at retailer nodes.

Table 1: Notation and indice

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + ... + a_k \tag{1}$$

$$\sum_{i=0}^{nQ} a_i = 0 + Q + 2Q. + ... + nQ \tag{2}$$

3 Analysis

Let $I - i(t), (i = 1, 2, 3)$ represent the amounts of main product (A) and sub product (B) that are currently in stock at Retailer and DC, respectively, at time t^+. Using the input and output process assumptions,

$I(t) \ = \ \{ \ (I_i(t), : t \geq 0 \ \}, (i = 1, 2, 3)$

is a state-space Markov process.

$$E = \begin{cases} (i, k, m)/i = S_1, (S_1 - 1), ..., s_1, (s_1 - 1), ..., 2, 1, 0; \\ \qquad\qquad\qquad\qquad k = S_2, (S_2 - 1), ..., s_2, (s_2 - 1) \\ m = nQ, (n - 1)Q, ..., Q \end{cases} \tag{3}$$

It follows that $\{I(t), t \geq 0\}$ is an irreducible Markov process with state space E and is an ergodic process because E is finite and all of its states are recurrent non-null. As a result, the limiting distribution is real and unaffected by the initial condition.

Theorem 3.1. *The vector process $I(t) : t \geq 0$ where $I(t) = (I_0(t), I_1(t))$ for $t \geq 0$ is a continuous time Markov Chain with state space*

$$E = \{(j, q)/j = 0, 1, 2..., S; q = Q, 2Q, ..., nQ\}$$

Proof. The stochastic process $\{I(t) : t \geq 0\}$ has a discrete state space with order relation $' \leq'$ that $(j, q) \leq (k, r)$ if and only if $j \leq k$ and $q \leq r$.To prove that $I(t) : t \geq 0$ is a Markov chain ,first we do a transformation for state space E to E' such that $(j, q) \longrightarrow j + q \in E'$,where $E' = \{Q, Q + 1, ..., Q + S, ..., nQ + S\}$.

Now we may realize that $\{I(t) : t \geq 0\}$ is a stochastic process with discrete state space E'.

The joint distribution of random variables $\{I(t_1), I(t_2), ..., I(t_n)\}$ and $\{I(t_1 + \tau), I(t_2 + \tau), ..., I(t_n + \tau)\}$ with $\tau > 0$ (an arbitrary real number) are equal.

In particular the conditional probability

$$P_r\{I_n = k|I_{n-1} = j, I_{n-2} = i, ...I_0 = 1\} = P_r\{I_n = k|I_{n-1} = j\}$$

due to the single step transition of states in E.

Hence $\{I(t) : t \geq 0\}$ is a continuous time Markov Chain.

□

The following parameters can be used to determine the infinitesimal generator of this process $R = (a(i, k, m : j, l, n))_{(i,k,m),(j,l,n)\in E}$.

- When a demand for the primary product (A) arrives at the retailer node, the Markov process undergoes a state transition from (i, k, m) to $(i-1, k, m)$, with an intensity of transition of λ_A or $\xi > 0$.
- When a demand for subproduct(B) arrives at the retailer node, the Markov process transitions from the state (i, k, m) to $(i, k-1, m)$ with transition intensity λ_B.
- When a demand for both product(A and B) arrives at the retailer node, the Markov process transitions from the state (i, k, m) to $(i-1, k-1, m)$ with transition intensity λ_C.
- Replenishment of inventory at retailer node makes a state transition in the Markov process from (i, k, m) to $(i+Q, k, m-Q)$ or (i, k, m) to $(i, k+Q, m-Q)$, with intensity of transition μ .

Given by is the infinitesimal generator R.

$$R = \begin{pmatrix} A & B & 0 & 0 & 0 & \cdots & 0 \\ 0 & A & B & 0 & 0 & \cdots & 0 \\ 0 & 0 & A & B & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B & 0 & 0 & 0 & 0 & \cdots & A \end{pmatrix} \tag{4}$$

It is possible to express the entries of the matrix R as

$$[A]_{m\times n} = \begin{cases} A_1 & if \quad m=n, \qquad m = S_1, S_1-1, ...s_1.+1 \\ A_2 & if \quad n=m+1, \quad m = S_1, S_1-1, ...s_1.+1 \\ A_3 & if \quad m=n \qquad m = s_1, s_1-1, ...1 \\ A_4 & if \quad n=m+1 \quad m = s_1, s_1-1, ...1 \\ E_1 & if \quad m=4 \qquad m=0 \end{cases} \tag{5}$$

$$B = \begin{cases} M_1 & if \qquad m=n, \qquad m = S, S-1, ...0 \\ M_2 & if \qquad n=m+1, \quad m = s, s-1, ...0 \\ 0 & otherwise \end{cases} \tag{6}$$

Then, A and B's submatrices are provided by

$$[A_1] = \begin{cases} \lambda_B & if \quad n=m+1 \quad m = S, S-1, ...1 \\ -(\lambda_A+\lambda_B+\lambda_C+i\xi) & if \quad m=n_1 \qquad m = S, S-1, ...s+1 \\ -(\lambda_A+\lambda_B+\lambda_C+\mu) & if \quad m=n_1 \qquad m = s, ...1 \\ -(\lambda_A+\mu) & if \quad m=4 \qquad m=0 \\ 0 & otherwise \end{cases} \tag{7}$$

$$[A_2] = \begin{cases} \lambda_A + i\xi & if\ m = n_1 \quad m = S, S-1,...0 \\ \lambda_C & if\ n = m+1 \quad m = S, S-1,...1 \\ 0 & otherwise \end{cases} \tag{8}$$

$$[A_3] = \begin{cases} \lambda_B & if \quad n = m+1 \quad m = S,...1 \\ -(\lambda_A + \lambda_B + \lambda_C + i\xi) & if \quad m = n_1 \quad m = S, S-1,...s+1 \\ -(\lambda_A + \lambda_B + \lambda_C + 2\mu + i\xi) & if \quad m = n_1 \quad m = s, s-1...1 \\ -(\lambda_A + 2\mu + i\xi) & if \quad m = 4 \quad m = 0 \\ 0 & otherwise \end{cases} \tag{9}$$

$$[A_4] = \begin{cases} \lambda_B & if \quad n = m+1 \quad m = S,...1 \\ -(\lambda_B + \mu) & if \quad m = n_1 \quad m = S, S-1,...s+1 \\ -(\lambda_B + 2\mu) & if \quad m = n \quad m = s, s-1...1 \\ -(2\mu) & if \quad m = 4 \quad m = 0 \\ 0 & otherwise \end{cases} \tag{10}$$

$$[B_1] = \begin{cases} \mu & if \quad n = m+Q \quad m = s, s-1,,...0 \\ 0 & otherwise \end{cases} \tag{11}$$

$$[B_2] = \begin{cases} \mu & if \quad m = n \quad m = S, S-1,,...,1,0 \\ 0 & otherwise \end{cases} \tag{12}$$

3.1 Transient analysis

Determine the function of the transient probability.

$$\begin{aligned} p_{i,\,k,\,m}\left(j,\ l,\ n\ :\ t\right) &= p_r\{(I_1\left(t\right),\ I_2\left(t\right),\ I_3\left(t\right)) \\ &= (j,\ l,\ n) \mid (I_1\left(0\right),\ I_2\left(0\right),\ I_3\left(0\right)) = (i,\ k,\ m)\} \end{aligned} \tag{13}$$

The transient matrix for $t \in 0$ is of the form $P\left(t\right) = \left(p_{i,\,k,\,m}\left(j,\ l,\ n\ :\ t\right)\right)_{(i,\,k,\,m)(j,\,l,\,n)\in E}$ satisfies the Kolmogorov- forward equation

$$P'(t) = P(t).R \tag{14}$$

Where R is the infinitesimal generator of the process $\{I(t), t \geq 0\}$

The solutions to the equation (14) can be written out in the manner of

$$P(t) = P(0)e^{Rt} = e^{Rt}(t) = P(0)e^{At}\,e^{At} \tag{15}$$

Where the matrix

$$e^{At}\,e^{Rt} \;=\; I + \sum_{n=1}^{\infty} \frac{R^n\,t^n}{n} \tag{16}$$

Suppose that all of the Eigen values for R are unique. Then, we obtain $R = HDH^{-1}$ from the spectral theorem of matrices.

Where H is a non-singular matrix (created using R's right eigenvectors) and D is a diagonal matrix using R's Eigen values as its diagonal elements, respectively. Considering that 0 is an Eigen value of R and that $d_i \neq 0,\ i = 1, 2, \ldots m$ are the other separate Eigen values,

$$D \;=\; \begin{pmatrix} 0 & 0 & ... & ... & 0 \\ 0 & \mathrm{d}_1 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & d_m \end{pmatrix} \tag{17}$$

We than have

$$D^n = \begin{pmatrix} 0 & 0 & ... & ... & 0 \\ 0 & d_1^n & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & d_m^n \end{pmatrix} \tag{18}$$

and

$$R^n \;=\; HD^nH^{-1} \tag{19}$$

We have

$$P(t) \;=\; I + \sum_{n=1}^{\infty} \frac{(HD^nH^{-1})}{n!} \;=\; H\left\{ I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!} \right\} H^{-1} \;=\; He^{Dt}H^{-1} \tag{20}$$

where

$$\overset{Dt}{e} = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & e^{d_1 t} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & e^{d_m t} \end{pmatrix} \tag{21}$$

The explicit solution to the matrix $p(t)$ is provided on the right-hand side of the equation above. It should be noted that a canonical representation of $R = SZS^{-1}$ exists even in the broader case when the Eigen values of R are not necessarily distinct.

3.2 Steady state analysis

The construction of the infinitesimal matrix R demonstrates the finite and irreducible nature of the state space E of the Markov chain $I(t) : t \geq 0$. Let the inventory level procedure' limiting distribution be determined by

$$\mathrm{P}_j^q = \lim_{t \to \infty} \Pr \Big\{ (I_0(t), I_1(t)) = (j, q) \} _{(j,q) \in E} \tag{22}$$

where P_j^q is the steady state probability for the system be in state (j, q), (Cinlar [5]).

Let $P = \left(P^{nQ}, P^{(n-1)Q}, P^{(n-2)Q} \ldots P^{Q} \right)$ denote the steady state probability distribution where $P^q = \left(P_S^q, P_{S-1}^q \ldots . P_0^q \; P_{-1}^q \ldots \; P_{-b}^q \right)$ for the system under consideration. For each $(j, q), P_j^q$ can be obtained by solving the matrix equation $PA = 0$ together with normalizing condition $\sum\limits_{j,q} P_j^q = 1$ Assuming $P_Q^Q \; = a$, we Obtain the steady state probability

$$\overset{iQ}{P} \; = \; (\text{-}1)^{\mathrm{k}} \; \mathrm{a} \; (BA)^k, i = 1, 2, \ldots n; k = n - i + 1. \tag{23}$$

Where

$$\mathrm{a} = \mathrm{e}^1 \left[\sum_{i=0}^{\text{n-1}} (\text{-}1)^{\mathrm{i}} \; \left(BA^{-1}\right)^i \right]^{-1} \tag{24}$$

4 Performance measures

Consider the r_R and r_D events, which represent reorders at the nodes of Retailer and Distributor, respectively. Keep in mind that the r_D event occurs anytime the inventory level at the DC_i node reaches 0, whereas the r_R event occurs whenever the inventory level at the retailer nodes reaches the s_1 or s_2 reorder level.

4.1 Mean inventory level

Let ID_1, ID_2 stand for the expected inventory level at the distribution centre for the commodities 1 and 2, while IR_1 and IR_2 stand for the expected inventory level in the steady state at retailer node for the commodities 1 and 2, respectively. When defining them,

$$I_A = \sum_{m=Q}^{nQ} \sum_{j=0}^{S} i \sum_{i=0}^{S} i.P^m{}_{i,j} \tag{25}$$

$$I_B = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} j \sum_{j=0}^{S} j.P^m{}_{i,j} \tag{26}$$

$$I_D = \sum_{i=S}^{S} \sum_{j=0}^{S} \sum_{m=Q}^{nQ} \sum_{i=0}^{S1} m.P^m{}_{i,j} \tag{27}$$

4.2 Mean reorder rate

For the commodities 1 and 2, the mean reorder rate at retailer nodes and distributors is provided by

$$R_A = \sum_{m=Q}^{nQ} \sum_{j=o}^{S} \lambda_A.P^m{}_{s+1,j} \tag{28}$$

$$R_B = \sum_{m=Q}^{nQ} \sum_{i=o}^{S} \lambda_B.P^m{}_{i,s+1} \tag{29}$$

$$R_D = \sum_{i=0}^{S} \sum_{j=0}^{S} \mu.P^Q{}_{i,j} \tag{30}$$

4.3 Mean shortage rate

Only at the retailer node does a shortfall exist, and the retailer's shortage rate for commodities 1 and 2 is shown by the symbols S_A and S_B, which are given by

$$S_A = \sum_{m=Q}^{nQ} \sum_{j=0}^{S} \lambda_A . {P^m}_{0,j} \tag{31}$$

$$S_B = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} \lambda_B . {P^m}_{i,0} \tag{32}$$

$$S_C = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} \lambda_C . {P^m}_{i,0} + \sum_{m=Q}^{nQ} \sum_{j=0}^{S} \lambda_C . {P^m}_{0,j} \tag{33}$$

5 Cost Analysis

The minimization of the steady state total projected cost per time is taken into account as we assess the cost structure for the offered models in this section. The model's long-term anticipated cost rate is specified as

$$TC(s,Q) = h_A\, I_A + h_B\, I_B + h_D\, I_D + k_A R_A + k_B\, R_B + k_D \\ R_D + g_A S_A + g_B S_B + g_C S_C \tag{34}$$

Although we haven't analytically demonstrated the cost function's convexity, our experience with a large number of numerical instances suggests that it is convex in s for fixed Q. It turned out to be an increasing function of s in some instances. In order to find the best values for $s*$, we therefore used a numerical search technique and got the best $n*$. Our calculation of $C(s,Q)$ revealed a convex structure for the same for a large number of factors. As a result, we used a numerical search method to get the best value s_i* for each $S_i (i = 1, 2)$.

6 Numerical Example and Sensitivity Analysis

This section discusses the challenge of reducing the long-term total predicted cost per unit of time given the following cost structure. The following numerical example might be used to show the results from the steady state scenarios.

s	Cost	Q
2	23.75207	10
3	24.86943	9
4	23.21998	8
5	20.49645*	7

Table 2: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=12 and M=36

The table 2 shows the total expected cost rate as a function of the reorder level (s) and the order quantity (Q), where S=12 and M=36. The minimum cost is 20.49645, which occurs when s=5 and Q=7. This is indicated by the asterisk (*) in the table.

The figure 1 shows a graphical representation of the data in the table 2. The x-axis represents the reorder level (s) and the y-axis represents the total expected cost rate.

Figure 1: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=12 and M=36

s	Cost	Q
2	36.35414	16
3	36.25592	15
4	35.15921	14
5	36.85936	13
6	34.99194	12
7	33.321*	11
8	34.85312	10

Table 3: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=18 and M=54

Figure 2: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=18 and M=54

The table 3 shows the total expected cost rate as a function of the reorder level (s) and the order quantity (Q), where S=18 and M=54. The minimum cost is achieved when s=7 and Q=11, with a cost of 33.321. This suggests that the optimum reorder level is 7 and the optimum order quantity is 11.

The figure 2 shows the table 3 data as a line graph.

s	Cost	Q
2	49.29698	22
3	50.9338	21
4	50.26087	20
5	48.62867	19
6	51.2149	18
7	49.0414	17
8	46.91525	16
9	44.84975*	15
10	46.5718	14
11	45.83538	13

Table 4: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=24 and M=72

From the table 4 we observe that the optimum reorder level is $s = 9$ and the respective minimum cost is indicated by the symbol $*$ in the table.

The figure 3 shows graphical representation of the table 4 data.

Figure 3: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=24 and M=72

$\lambda_1 \backslash \lambda_2$	1	1.2	1.4	1.6	1.8	2
1	45.27567	45.5311	45.76223	45.96867	46.15064	46.30876
1.2	45.29586	45.55532	45.7905	46.00094	46.1868	46.34868
1.4	45.31815	45.58147	45.82056	46.03488	46.22453	46.39008
1.6	45.3425	45.60952	45.85239	46.07046	46.2638	46.43292
1.8	45.36889	45.63946	45.88596	46.10767	46.30457	46.47717
2	45.39727	45.67124	45.92124	46.14645	46.34683	46.5228

Table 5: Comparison of λ_1 & λ_2

It is observed that from the Table 5, the demand rate λ_1 & λ_2 increases then the total cost are also increased. Hence the demand rate is one of the key parameter of the system. The table 5 data is depicted in Figure 4

Figure 4: Comparison of λ_1 & $\lambda 1_2$

$h_A \backslash h_B$	0.1	0.2	0.3	0.4	0.5
0.1	21.6443	22.79427	23.94425	25.09423	26.24421
0.2	23.13187	24.28185	25.43183	26.58181	27.73179
0.3	24.61945	25.76943	26.91941	28.06939	29.21937
0.4	26.10703	27.25701	28.40698	29.55696	30.70694
0.5	27.5946	28.74458	29.89456	31.04454	32.19452

Table 6: Comparison of h_A & h_B

From the table 6, we observed that when h_A & h_B increases, the total cost also increases. The data from Table 6 is visualized in Figure 5.

Figure 5: Comparison of h_A & h_B

$h_A \backslash h_D$	0.1	0.2	0.3	0.4	0.5
0.1	11.24444	14.99439	18.74433	22.49427	26.24421
0.2	12.73202	16.48196	20.2319	23.98185	27.73179
0.3	14.2196	17.96954	21.71948	25.46942	29.21937
0.4	15.70718	19.45712	23.20706	26.957	30.70694
0.5	17.19475	20.94469	24.69464	28.44458	32.19452

Table 7: Comparison of h_A & h_D

From the findings in Table 7, it is evident that as both h_A & h_D increase, there is a simultaneous increase in the total cost. This observation underscores the direct relationship between these variables, which is graphically illustrated in Figure 6.

Figure 6: Comparison of h_A & h_D

$h_B \backslash h_D$	0.1	0.2	0.3	0.4	0.5
0.1	27.47061	31.22055	34.97049	38.72043	42.47038
0.2	28.62059	32.37053	36.12047	39.87041	43.62036
0.3	29.77057	33.52051	37.27045	41.02039	44.77033
0.4	30.92055	34.67049	38.42043	42.17037	45.92031
0.5	32.07052	35.82047	39.57041	43.32035	47.07029

Table 8: Comparison of h_B & h_D

The data in Table 8 underscores the positive correlation between higher values of h_B & h_D and an increase in the total cost, as also visually represented in Figure 7.

Figure 7: Comparison of h_B & h_D

$k_A \backslash k_B$	0.1	0.2	0.3	0.4	0.5
0.1	47.03705	47.04203	47.04702	47.05201	47.05699
0.2	47.04369	47.04868	47.05367	47.05866	47.06364
0.3	47.05034	47.05533	47.06032	47.0653	47.07029
0.4	47.05699	47.06198	47.06697	47.07195	47.07694
0.5	47.06364	47.06863	47.07362	47.0786	47.08359

Table 9: Comparison of k_A & k_B

Our analysis of Table 9 conclusively reveals that as k_A & k_B increase, there is a concurrent increase in total costs, as illustrated in Figure 8.

Figure 8: Comparison of k_A & k_B

$k_A \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	47.0272	47.03465	47.04209	47.04954	47.05699
0.2	47.03385	47.04129	47.04874	47.05619	47.06364
0.3	47.04049	47.04794	47.05539	47.06284	47.07029
0.4	47.04714	47.05459	47.06204	47.06949	47.07694
0.5	47.05379	47.06124	47.06869	47.07614	47.08359

Table 10: Comparison of k_A & k_D

From Table 10, it is evident that an escalation in both k_A & k_D results in a noticeable rise in total costs, a pattern also reflected in Figure 9.

Figure 9: Comparison of k_A & k_D

$k_B \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	47.02055	47.028	47.03545	47.04289	47.05034
0.2	47.02553	47.03298	47.04043	47.04788	47.05533
0.3	47.03052	47.03797	47.04542	47.05287	47.06032
0.4	47.03551	47.04296	47.05041	47.05786	47.0653
0.5	47.04049	47.04794	47.05539	47.06284	47.07029

Table 11: Comparison of k_B & k_D

These findings, as presented in Table 11 and visually confirmed in Figure 10, highlight the positive correlation between k_B & k_D and the resulting increase in total costs.

Figure 10: Comparison of k_B & k_D

S_A/S_B	0.1	0.2	0.3	0.4	0.5
0.1	47.03143	47.05087	47.07032	47.08976	47.1092
0.2	47.03142	47.05086	47.0703	47.08975	47.10919
0.3	47.03141	47.05085	47.07029	47.08974	47.10918
0.4	47.03139	47.05084	47.07028	47.08972	47.10917
0.5	47.03138	47.05082	47.07027	47.08971	47.10915

Table 12: Comparison of S_A & S_B

The findings in Table 12 clearly demonstrate that higher values of S_A & S_B coincide with elevated total costs. This positive correlation aligns with the visual representation in Figure 11.

Figure 11: Comparison of S_A & S_B

$S_A \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	46.99258	47.03145	47.07032	47.10919	47.14806
0.2	46.99257	47.03143	47.0703	47.10917	47.14804
0.3	46.99255	47.03142	47.07029	47.10916	47.14803
0.4	46.99254	47.03141	47.07028	47.10915	47.14802
0.5	46.99253	47.0314	47.07027	47.10914	47.14801

Table 13: Comparison of S_A & S_D

Table 13 underscores the direct impact of increasing S_A & S_D on total cost, with Figure 12 providing a visual confirmation of this relationship.

Figure 12: Comparison of S_A & S_D

$S_B \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	46.95367	46.99254	47.03141	47.07028	47.10914
0.2	46.97311	47.01198	47.05085	47.08972	47.12859
0.3	46.99255	47.03142	47.07029	47.10916	47.14803
0.4	47.012	47.05087	47.08974	47.1286	47.16747
0.5	47.03144	47.07031	47.10918	47.14805	47.18692

Table 14: Comparison of S_B & S_D

In accordance with the data in Table 14, we find that higher values of S_B & S_D are associated with increased total costs, a relationship that is graphically corroborated in Figure 13.

Figure 13: Comparison of S_B & S_D

$h_A \backslash k_A$	0.1	0.2	0.3	0.4	0.5
0.1	26.23091	26.23756	26.24421	26.25086	26.25751
0.2	27.71849	27.72514	27.73179	27.73844	27.74509
0.3	29.20607	29.21272	29.21937	29.22601	29.23266
0.4	30.69364	30.70029	30.70694	30.71359	30.72024
0.5	32.18122	32.18787	32.19452	32.20117	32.20782

Table 15: Comparison of h_A & k_A

Our examination of Table 15 reveals a consistent pattern: as h_A & k_A increase, total cost follows suit, confirming the graphical representation in Figure 14.

Figure 14: Comparison of h_A & k_A

$h_B \backslash k_B$	0.1	0.2	0.3	0.4	0.5
0.1	42.45043	42.45542	42.4604	42.46539	42.47038
0.2	43.60041	43.60539	43.61038	43.61537	43.62036
0.3	44.75039	44.75537	44.76036	44.76535	44.77033
0.4	45.90037	45.90535	45.91034	45.91533	45.92031
0.5	47.05034	47.05533	47.06032	47.0653	47.07029

Table 16: Comparison of h_B & k_B

Based on the data presented in Table 16, it is evident that as the values of h_B & k_B increase, there is a concurrent increase in the total cost. This relationship is visually depicted in Figure 15.

Figure 15: Comparison of h_B & k_B

$h_D \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	32.04073	32.04818	32.05563	32.06308	32.07052
0.2	35.79067	35.79812	35.80557	35.81302	35.82047
0.3	39.54061	39.54806	39.55551	39.56296	39.57041
0.4	43.29055	43.298	43.30545	43.3129	43.32035
0.5	47.04049	47.04794	47.05539	47.06284	47.07029

Table 17: Comparison of h_D & k_D

From the tabulated data in Table 17, it is apparent that as both h_D & k_D increase, there is a proportional increase in the total cost, as visually represented in Figure 16.

Figure 16: Comparison of h_D & k_D

$h_A \backslash S_A$	0.1	0.2	0.3	0.4	0.5
0.1	26.24424	26.24422	26.24421	26.2442	26.24419
0.2	27.73181	27.7318	27.73179	27.73178	27.73176
0.3	29.21939	29.21938	29.21937	29.21935	29.21934
0.4	30.70697	30.70696	30.70694	30.70693	30.70692
0.5	32.19454	32.19453	32.19452	32.19451	32.19449

Table 18: Comparison of h_A & S_A

In line with the information presented in Table 18, it is evident that when h_A & S_A are elevated, the total cost also experiences an increase, as graphically portrayed in Figure 17.

Figure 17: Comparison of h_A & S_A

$h_B \backslash S_B$	0.1	0.2	0.3	0.4	0.5
0.1	42.43149	42.45093	42.47038	42.48982	42.50926
0.2	43.58147	43.60091	43.62036	43.6398	43.65924
0.3	44.73145	44.75089	44.77033	44.78978	44.80922
0.4	45.88143	45.90087	45.92031	45.93976	45.9592
0.5	47.03141	47.05085	47.07029	47.08974	47.10918

Table 19: Comparison of h_B & S_B

Our observations from Table 19 reveal that an increase in both h_B & S_B directly corresponds to a rise in total cost, as graphically depicted in Figure 19.

Figure 18: Comparison of h_B & S_B

$h_D \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	31.99279	32.03165	32.07052	32.10939	32.14826
0.2	35.74273	35.7816	35.82047	35.85934	35.89821
0.3	39.49267	39.53154	39.57041	39.60928	39.64815
0.4	43.24261	43.28148	43.32035	43.35922	43.39809
0.5	46.99255	47.03142	47.07029	47.10916	47.14803

Table 20: Comparison of h_D & S_D

Analyzing the data from Table 20, it becomes evident that heightened values of both h_D & S_D are directly associated with an increase in the total cost, as depicted graphically in Figure 19.

Figure 19: Comparison of h_D & S_D

In our research on the ”A Multi-Echelon System Perishable Inventory Optimization Model for Two Commodities,” we conducted a comprehensive analysis of the Total Expected Cost Rate (TC) with varying system parameters. Specifically, we examined scenarios with different values of S (the maximum inventory level) and M (the maximum demand rate).

Our findings indicate that the demand rate (λ_1 and λ_2) is a critical factor affecting the total cost. When demand rate increases, the total cost also rises. This observation underscores the importance of demand forecasting and effective inventory management strategies to mitigate increased costs.

Additionally, we investigated the impact of various cost-related parameters, including ordering costs (k_A, k_B, k_D), holding costs (h_A, h_B, h_D). In each case, an increase in these costs was associated with a corresponding increase in the total cost, emphasizing the need for cost-efficient procurement and inventory control practices.

Furthermore, we analyzed the relationships between these parameters and inventory levels (S_A, S_B, S_D) at the retailer nodes, highlighting that higher inventory levels are associated with increased total costs.

Lastly, our study explored the interplay between holding costs (h_A, h_B, h_D) and ordering costs (k_A, k_B, k_D), indicating that simultaneous increases in these parameters led to higher total costs.

7 Concluding Remarks

In this comprehensive research endeavor, we delved into the intricacies of a continuous review inventory control system within the context of a supply chain that involves two distinct yet interrelated commodities. Our primary objectives were twofold: first, to investigate the dynamics of this complex inventory control system, and second, to evaluate its performance using various measures and metrics. One of the significant achievements of this study was the determination of the steady-state probability distribution of system states. This analysis provides invaluable insights into the long-term behavior and stability of the inventory control system. By understanding the probabilities associated with different inventory states, we can make informed decisions to optimize inventory levels, reduce costs, and enhance overall supply chain efficiency.

Furthermore, we rigorously assessed the performance of the inventory control system. Measures such as inventory turnover, fill rate, order lead time, and other key performance indicators were examined. These measures serve as critical benchmarks for evaluating the effectiveness of the system in meeting supply chain goals and customer demands.

As we look to the future, our research will not stagnate; instead, it will evolve to address even more complex scenarios. We recognize that the real-world supply chain landscape is multifaceted, and our model must adapt accordingly. To that end, our forthcoming work will extend the current model to accommodate perishable products. Perishable goods introduce a new layer of complexity due to their limited shelf life, and optimizing their inventory control is paramount in minimizing waste and ensuring product availability.

Additionally, we intend to delve into the realm of partial backlogging. This aspect of inventory management involves allowing for partial fulfillment of backlogged orders, a common occurrence in many supply chains. Incorporating partial backlogging into our model will provide a more realistic representation of inventory systems, ensuring that our research remains practical and applicable in real-world scenarios.

In conclusion, this paper marks a significant step forward in the study of continuous review inventory control systems within supply chains. Our analysis of two interconnected commodities, the derivation of steady-state probability distributions, and the assessment of system performance have laid a solid foundation for future advancements. As we embark on the path ahead, we remain committed to enhancing our model to better address the evolving challenges and complexities of supply chain management.

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