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Anti-Adjacency Matrices of Certain Graphs Derived from Some Graph Operations

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Abstract

If we go through the literature, one can find many matrices that are derived for a given simple graph. The one among them is the anti-adjacency matrix which is given as follows; The anti-adjacency matrix of a simple undirected graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ matrix $B(G) = (b_{ij})$, where $b_{ij} = 0$ if there exists an edge between v_i and v_j and 1 otherwise. In this paper, we try to bring out an expression, which establishes a connection between the anti-adjacency matrices of the two graphs G_1 and G_2 and the anti-adjacency matrix of their tensor product, $G_1 \otimes G_2$. In addition, an expression for the anti-adjacency matrix of the disjunction of two graphs, $G_1 \vee G_2$, is obtained in a similar way. Finally, we obtain an expression for the anti-adjacency matrix for the generalized tensor product and generalized disjunction of two graphs. Adjacency and anti-adjacency matrices are square matrices that are used to represent a finite graph in graph theory and computer science. The matrix elements show whether a pair of vertices in the graph are adjacent or not.

Keywords: Anti-adjacency matrix, tensor product, disjunction, generalized tensor product and generalized disjunction.
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1 Introduction

The tensor product (conjunction) and disjunction of two graphs were first described in 1967 by Frank Harary and Gordon W. Wilcox [2]. The idea of the generalized tensor product of two graphs was first suggested in 2014 by U P Acharya and H S Mehta [5]. The concept of generalized disjunction of two graphs with respect to distance as a parameter was first suggested in 2020 by Manju V N and G Suresh Singh [4]. M. Edwina and K.A. Sugeng established the concept of anti-adjacency matrix for undirected graphs in 2017[1]. Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. According to [1], the anti-adjacency matrix of G, denoted by $B(G)$ is the $n \times n$ matrix $B = (b_{ij})$, where $b_{ij} = 0$ if $v_i v_j \in E(G)$ or otherwise $b_{ij} = 1$. Some properties of anti-adjacency matrix are related to its characteristic polynomial and its eigenvalues. In chemical graph theory, the physical characteristics of chemical graphs can be obtained by using the eigenvalues of the anti-adjacency matrix of graphs. This motivates us to compute the anti-adjacency matrix of some graphs obtained by few graph theoretical operations.

In this paper, we first obtain an expression for the anti-adjacency matrix of tensor product and disjunction of two graphs. Next, we define the r^{th} stage anti-adjacency matrix of a simple undirected graph and then try to get an expression for the anti-adjacency matrix for the generalized tensor product and generalized disjunction of two graphs.

Definition 1.1. *[3] The distance between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$ is the length of the shortest path between u and v in G.*

Definition 1.2. *[2] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The tensor product (or conjunction) of G_1 and G_2, denoted by $G_1 \otimes G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes G_2$ if u is adjacent to u' in G_1 and v is adjacent to v' in G_2.*

Definition 1.3. *[5] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The m-tensor product of G_1 and G_2, denoted by $G_1 \otimes_m G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes_m G_2$ if $d_{G_1}(u, u') = m$ and $d_{G_2}(v, v') = m$.*

It is clear that when $m = 1$, we get definition 1.2. When $m = 2$, we have the following

Definition 1.4. *[5] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The 2-tensor product of G_1 and G_2, denoted by $G_1 \otimes_2 G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes_2 G_2$ if $d_{G_1}(u, u') = 2$ and $d_{G_2}(v, v') = 2$.*

Definition 1.5. *[2] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The disjunction of G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee G_2$ if u is adjacent to u' in G_1 or v is adjacent to v' in G_2 or both*

Definition 1.6. *[4] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The $m-$ disjunction of G_1 and G_2, denoted by $G_1 \vee_m G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee_m G_2$ if $d_{G_1}(u, u') = m$ or $d_{G_2}(v, v') = m$ or both.*

It is clear that when $m = 1$, we get definition 1.5. When $m = 2$, we have

Definition 1.7. *[4] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The $2-$ disjunction of G_1 and G_2, denoted by $G_1 \vee_2 G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee_2 G_2$ if $d_{G_1}(u, u') = 2$ or $d_{G_2}(v, v') = 2$ or both.*

Definition 1.8. *[1] Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The anti-adjacency matrix of G, denoted by $B(G)$ is the $n \times n$ matrix $B = [b_{ij}]$, where $b_{ij} = 0$ if $e = v_i v_j \in E(G)$ or otherwise $b_{ij} = 1$.*

Definition 1.9. *[2] Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix, then their tensor product denoted by $A \otimes B$ is an $mp \times nq$ matrix and is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

All graphs under our consideration are simple, finite undirected connected graphs.

2 Anti-Adjacency Matrix of $m-$ Tensor Product of Graphs

In this section we study some properties of the anti-adjacency matrix of $G_1 \otimes G_2$ and generalize the result for m- tensor product of G_1 and G_2.

Theorem 2.1. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes G_2$ is given by $B(G_1 \otimes G_2) = B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2 and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \otimes G_2)$,

$$B(G_1 \otimes G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle ...} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) \neq 1$, then $P_{ij} = 1 \; \forall \; i$ and j. So, in this case $B_{rs} = J_n$.
Next suppose that $d_{G_1}(u_r, u_s) = 1$, then $P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 1 \\ 1 & \text{otherwise} \end{cases}$
Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B(G_2)$. Hence,

$$B_{rs} = \begin{cases} B(G_2) & \text{if } d_{G_1}(u_r, u_s) = 1 \\ J_n & \text{otherwise} \end{cases}$$

Assume, $B(G_1) = [b_{ij}]_{m \times m}$ and $B(G_2) = [c_{ij}]_{n \times n}$ and consider

$$B(G_1) \otimes J_n = \begin{bmatrix} b_{11}J_n & b_{12}J_n & \cdots & b_{1m}J_n \\ b_{21}J_n & b_{22}J_n & \cdots & b_{2m}J_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}J_n & b_{m2}J_n & \cdots & b_{mm}J_n \end{bmatrix}$$

Then the $(i, j)^{th}$ block of $B(G_1) \otimes J_n = [b_{ij}J_n]$ and

$$[b_{ij}J_n] = \begin{cases} (0)_{n \times n} & \text{if } b_{ij} = 0 \text{ (ie, if} d_{G_1}(u_i, u_j) = 1) \\ J_n & \text{otherwise} \end{cases}$$

Now,

$$J_m \otimes B(G_2) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes B(G_2)$$
$$= \begin{bmatrix} B(G_2) & B(G_2) & \cdots & B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ B(G_2) & B(G_2) & \cdots & B(G_2) \end{bmatrix}$$

So, the $(i,j)^{th}$ block of $J_m \otimes B(G_2)$ is $B(G_2)\ \forall\ i$ and j. Next consider,

$$B(G_1) \otimes B(G_2) = \begin{bmatrix} b_{11}B(G_2) & b_{12}B(G_2) & \cdots & b_{1m}B(G_2) \\ b_{21}B(G_2) & b_{22}B(G_2) & \cdots & b_{2m}B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B(G_2) & b_{m2}B(G_2) & \cdots & b_{mm}B(G_2) \end{bmatrix}$$

Hence, the $(i,j)^{th}$ block of $B(G_1) \otimes B(G_2)$ is $b_{ij}B(G_2)$ and

$$b_{ij}B(G_2) = \begin{cases} (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 1) \\ B(G_2) & \text{if } b_{ij} = 1 \text{ (ie, if } d_{G_1}(u_i, u_j) \neq 1) \end{cases}$$

It follows that the $(i,j)^{th}$ block of $B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$

$$= \begin{cases} (0)_{n\times n} \oplus B(G_2) \oplus (0)_{n\times n} & \text{if } d_{G_1}(u_i, u_j) = 1 \\ J_n \oplus B(G_2) \oplus B(G_2) & \text{if } d_{G_1}(u_i, u_j) \neq 1 \end{cases}$$
$$= \begin{cases} B(G_2) & \text{if } d_{G_1}(u_i, u_j) = 1 \\ J_n & \text{if } d_{G_1}(u_i, u_j) \neq 1 \end{cases}$$

Therefore, the $(i,j)^{th}$ block of $B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \otimes G_2)$. Hence, $B(G_1 \otimes G_2) = B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$. □

Definition 2.1. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The second stage anti-adjacency matrix of G, $B_2(G)$ is the $n \times n$ matrix $B_2(G) = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 0 & \textit{if } d_G(v_i, v_j) = 2 \\ 1 & \textit{otherwise} \end{cases}.$$

Note that $B_2(G)$ is real, square symmetric matrix.

Theorem 2.2. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes_2 G_2$ is given by $B(G_1 \otimes_2 G_2) = B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2 and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \otimes_2 G_2)$,

$$B(G_1 \otimes_2 G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle \cdots} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) \neq 2$, then $P_{ij} = 1 \; \forall \; i$ and j. So, in this case $B_{rs} = J_n$.

Next suppose that $d_{G_1}(u_r, u_s) = 2$, then $P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 2 \\ 1 & \text{otherwise} \end{cases}$

Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B_2(G_2)$. Hence,

$$B_{rs} = \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_r, u_s) = 2 \\ J_n & \text{otherwise} \end{cases}$$

Assume, $B_2(G_1) = [b_{ij}]_{m \times m}$ and $B_2(G_2) = [c_{ij}]_{n \times n}$ and consider

$$B_2(G_1) \otimes J_n = \begin{bmatrix} b_{11}J_n & b_{12}J_n & \cdots & b_{1m}J_n \\ b_{21}J_n & b_{22}J_n & \cdots & b_{2m}J_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}J_n & b_{m2}J_n & \cdots & b_{mm}J_n \end{bmatrix}$$

Then the $(i, j)^{th}$ block of $B_2(G_1) \otimes J_n = [b_{ij}J_n]$ and

$$[b_{ij}J_n] = \begin{cases} (0)_{n \times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 2) \\ J_n & \text{otherwise} \end{cases}$$

Now,

$$J_m \otimes B_2(G_2) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes B_2(G_2)$$

$$= \begin{bmatrix} B_2(G_2) & B_2(G_2) & \cdots & B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_2(G_2) & B_2(G_2) & \cdots & B_2(G_2) \end{bmatrix}$$

So, the $(i,j)^{th}$ block of $J_m \otimes B_2(G_2)$ is $B_2(G_2)\ \forall\ i$ and j. Next consider,

$$B_2(G_1) \otimes B_2(G_2) = \begin{bmatrix} b_{11}B_2(G_2) & b_{12}B_2(G_2) & \cdots & b_{1m}B_2(G_2) \\ b_{21}B_2(G_2) & b_{22}B_2(G_2) & \cdots & b_{2m}B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B_2(G_2) & b_{m2}B_2(G_2) & \cdots & b_{mm}B_2(G_2) \end{bmatrix}$$

Hence, the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2)$ is $b_{ij}B_2(G_2)$ and

$$b_{ij}B_2(G_2) = \begin{cases} (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i,u_j) = 2) \\ B_2(G_2) & \text{if } b_{ij} = 1 \text{ (ie, if } d_{G_1}(u_i,u_j) \neq 2) \end{cases}$$

It follows that the $(i,j)^{th}$ block of $B_2(G_1)\otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1)\otimes B_2(G_2)$

$$= \begin{cases} (0)_{n\times n} \oplus B_2(G_2) \oplus (0)_{n\times n} & \text{if } d_{G_1}(u_i,u_j) = 2 \\ J_n \oplus B_2(G_2) \oplus B_2(G_2) & \text{if } d_{G_1}(u_i,u_j) \neq 2 \end{cases}$$

$$= \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_i,u_j) = 2 \\ J_n & \text{if } d_{G_1}(u_i,u_j) \neq 2 \end{cases}$$

Therefore, the $(i,j)^{th}$ block of $B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \otimes_2 G_2)$. Hence, $B(G_1 \otimes_2 G_2) = B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$. □

Definition 2.2. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The r^{th} stage anti-adjacency matrix of G, $B_r(G)$ is the $n \times n$ matrix $B_r(G) = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 0 & \textit{if } d_G(v_i,v_j) = r \\ 1 & \textit{otherwise} \end{cases}.$$

Theorem 2.3. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes_m G_2$ is given by $B(G_1 \otimes_m G_2) = B_m(G_1) \otimes J_n \oplus J_m \otimes B_m(G_2) \oplus B_m(G_1) \otimes B_m(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2, $\otimes$ denotes the tensor product and $B_m(G)$ is the m^{th} stage anti-adjacency matrix of G.*

Proof. Proof is same as the proof of Theorem 2.2. □

3 Anti-adjacency Matrix of $m-$ Disjunction of G_1 and G_2

In this section we study some properties of the anti-adjacency matrix of $G_1 \vee G_2$ and generalize the result for m- disjunction G_1 and G_2.

Theorem 3.1. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee G_2$ is given by $B(G_1 \vee G_2) = B(G_1) \otimes B(G_2)$, where $B(G)$ is the anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \vee G_2)$,

$$B(G_1 \vee G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle \cdots} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) = 1$, then $P_{ij} = 0 \; \forall \; i$ and j. So, in this case B_{rs} is the zero matrix of order $n \times n$.
Next suppose that $d_{G_1}(u_r, u_s) \neq 1$, then

$$P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 1 \\ 1 & \text{otherwise} \end{cases}$$

Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B(G_2)$. Hence,

$$B_{rs} = \begin{cases} B(G_2) & \text{if } d_{G_1}(u_r, u_s) \neq 1 \\ (0)_{n \times n} & \text{otherwise} \end{cases}$$

Assume, $B(G_1) = [b_{ij}]_{m\times m}$ and $B(G_2) = [c_{ij}]_{n\times n}$ and consider

$$B(G_1) \otimes B(G_2) = \begin{bmatrix} b_{11}B(G_2) & b_{12}B(G_2) & \cdots & b_{1m}B(G_2) \\ b_{21}B(G_2) & b_{22}B(G_2) & \cdots & b_{2m}B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B(G_2) & b_{m2}B(G_2) & \cdots & b_{mm}B(G_2) \end{bmatrix}$$

Then the $(i,j)^{th}$ block of $B(G_1) \otimes B(G_2) = [b_{ij}B(G_2)]$ and

$$[b_{ij}B(G_2)] = \begin{cases} B(G_2) & \text{if } b_{ij} = 1\text{(ie, if } d_{G_1}(u_i, u_j) \neq 1) \\ (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 1) \end{cases}$$

Hence, the $(i,j)^{th}$ block of $B(G_1)\otimes B(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \vee G_2)$.
Therefore, $B(G_1 \vee G_2) = B(G_1) \otimes B(G_2)$.

□

Theorem 3.2. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee_2 G_2$ is given by $B(G_1 \vee_2 G_2) = B_2(G_1) \otimes B_2(G_2)$, where $B_2(G)$ is the second stage anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \vee_2 G_2)$,

$$B(G_1 \vee_2 G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathbf{B}_{rs} = \begin{array}{c} \\ (u_r,v_1) \\ \vdots \\ (u_r,v_n) \end{array} \begin{array}{c} \begin{array}{cccc} (u_s,v_1) & (u_s,v_2) & ... & (u_s,v_n) \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) = 2$, then $P_{ij} = 0 \ \forall \ i$ and j. So, in this case B_{rs} is the zero matrix of order $n \times n$.
Next suppose that $d_{G_1}(u_r, u_s) \neq 2$, then

$$P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 2 \\ 1 & \text{otherwise} \end{cases}$$

Thus in this case, $P_{ij} = (i,j)^{th}$ entry of $B_2(G_2)$. Hence,

$$B_{rs} = \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_r, u_s) \neq 2 \\ (0)_{n\times n} & \text{otherwise} \end{cases}$$

Assume, $B_2(G_1) = [b_{ij}]_{m\times m}$ and $B_2(G_2) = [c_{ij}]_{n\times n}$ and consider

$$B_2(G_1) \otimes B_2(G_2) = \begin{bmatrix} b_{11}B_2(G_2) & b_{12}B_2(G_2) & \cdots & b_{1m}B_2(G_2) \\ b_{21}B_2(G_2) & b_{22}B_2(G_2) & \cdots & b_{2m}B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B_2(G_2) & b_{m2}B_2(G_2) & \cdots & b_{mm}B_2(G_2) \end{bmatrix}$$

Then the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2) = [b_{ij}B_2(G_2)]$ and

$$[b_{ij}B_2(G_2)] = \begin{cases} B_2(G_2) & \text{if } b_{ij} = 1 \text{(ie, if } d_{G_1}(u_i, u_j) \neq 2) \\ (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 2) \end{cases}$$

Hence, the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \vee_2 G_2)$.
Therefore, $B(G_1 \vee_2 G_2) = B_2(G_1) \otimes B_2(G_2)$.

□

Theorem 3.3. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee_m G_2$ is given by $B(G_1 \vee_m G_2) = B_m(G_1) \otimes B_m(G_2)$, where $B_m(G)$ is the m^{th} stage anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. The Proof is same as the proof of Theorem 3.2. □

4 Conclusions

In this paper, we found the relation between the anti-adjacency matrices $B(G_1 \otimes G_2)$, $B(G_1)$ and $B(G_2)$. Also, we obtained the relationship between $B(G_1 \vee G_2), B(G_1)$ and $B(G_2)$. Further, we introduced the concept of r^{th} stage anti-adjacency matrix of a simple undirected graph and then tried to find the anti-adjacency matrices of graphs derived from the graph operations generalized tensor product and generalized disjunction. We strongly believe that this work can be further extended to find the anti-adjacency matrices of graphs, derived from other graph theoretical operations such as composition, Cartesian product etc.

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