Ratio Mathematica Volume 48, 2023

Optimization of fuzzy two-level production inventory system with persuasive exertion-reliant demand

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Abstract

Dynamic manufacturing-inventory system's intelligent production design develops a crucial problem in the business reactivity of indecisions. This investigation examines a two-level production system under fuzzy parameters and decision variables by implementing a pentagonal fuzzy quantity. As designed, the fuzzy system is defuzzified through the graded mean technique, and then the Kuhn-Tucker technique is utilised to obtain the optimum production size and shortage level. The effectual algorithms are established to project an intellectual industrial approach such as optimal production quantity, shortage level, trade group's exertions, and then minimum integrated expected total cost. The evaluation of the proposed fuzzy system is primed to fit numerical examples compared to the crisp system. This suggested fuzzy system is likewise compared with a particular case of the prior model. Numerical illustrations and sensitiveness studies are delivered in the direction of exhibiting the applicability based on the offered procedure and the attained outcomes.

Keywords: Optimal production; Shortage level; Trade group's exertions demand; Pentagonal fuzzy quantity; Kuhn-Tucker method.

2010 AMS subject classification: 94D05, 35Q93.

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‡ Received on March 9, 2023. Accepted on August 10, 2023. Published on September 3, 2023. doi: 10.23755/rm.v48i0.1152. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

An old-style isolated tactic for commercial industries does not work in the current violent and forceful international commercial situation. Advanced tactics enable revenue enhancements, price reductions, and tractability when addressing commercial indecisions. Through supply chain management, that chance can be realised through cooperative associations that exploit anticipated facility stages, decrease inventory prices, and create customer advantages. Additionally, current inventory models presume that all products are good. A specific ratio of products manufactured is in poor condition, regardless of the industry. In order to address the problem of imperfect quality, studies are currently being conducted to improve the inventory model.

Zadeh [13] presented the idea of fuzzy collections as an attempt at developing a set of concepts and techniques for dealing in a systematic way with imprecision objects that are not sharply defined. Dubois and Prade [3] devoted a structured exposition to system-oriented, realistic topics dealing with the fuzzy situation. Zimmermann [7] presented the fuzzy set concept, and its presentation part initially stretches the fundamentals of uncertain mathematics, while the next is concerned with applications of fuzzy set theory. Chen [19] presented a relationship function and the arithmetic operations of fuzzy numbers. Chang et. al. [18] considered the fuzzy state in the traditional inventory by backorder model due to some tangible circumstances. It is simple to fuzzify q, s or together. The common fuzzy number's graded mean defuzzification method was initially hosted by Chen and Hseih [23]. Chen et al. [24] extended the comprehensive L-R type fuzzy quantity through graded mean integration depiction and proved certain ranking possessions. Goyal and Cardenas-Barron [20] dealt with an inventory typical that is framed with a fuzzy differential equation, then the parallel inventory expenditures are also determined by fuzzy Riemann-integration. Yao and Chiang [11] investigated how to obtain the minimum entire price and the optimum order size through defuzzifying the whole price over the centroid process and the same mechanism executed via the signed distance process. Chiang et al. [9] proposed an inventory system to estimate the whole price in the ambiguous logic. Two circumstances of the results are fuzzy loading volume and order volume by way of triangular fuzzy quantities used numerically through the Nedler-Mead procedure.

Kazemi et al. [14] developed an inventory system for shortages in an uncertain environment by hiring binary categories of fuzzy quantities, which are triangular and trapezoidal. The prime solution for the established typical is calculated via the Kuhn-Tucker technique. An inventory typical designed aimed at objects using damaged levels, lack of backorder over fuzzy situations, was investigated by Mahata and Goswami [5]. Guchhait et al. [15] developed an integrating uncertain production rate in flawed production practise. A fuzzy inventory, typical of declining things along with time-oscillating requests, lacks under fully backlogged conditions, as framed by Harish Nagar and Priyanka Surana [8]. The Graded mean technique exists to utilise defuzzifying entire price tasks and the outcomes determined through this method.

Priyan et al. [22] assumed demand rate is depends on sensitive to persuasive actions or trade group initiatives. The inventory system considers manufacture levels as uncertain, but conceivable to define by fuzzy triangular quantity. Kamble [10] discussed the idea of pentagonal uncertain numbers. Established pentagonal fuzzy quantities are considered by the resources of internal math operations by consuming alpha-cut processes. Chakraborty et. al. [1] explored nonlinear membership roles for observing both symmetric and asymmetric pentagonal fuzzy systems. Vithyadevi and Annadurai [17] derived a combined inventory system through gathering cost decline reliant on lead interval in an uncertain situation by hiring a trapezoidal fuzzy number. The optimum result used for the established model is calculated by the Lagrangian method.

Taha [6], Hillier and Lieberman [4] gave uncertain problem-solving methods in operation research. Mathematical programming and stochastic models are introduced in operations research. Kaufmann and Gupta [2] delivered an introduction to fuzzy numbers and the procedures usage and gave the basic definitions and operations with many examples. Cardenas-Barron et al.'s [12] derived inventory manufacture system designed to dual-level transfer sequence involving one manufacturer and one vendor with the demand is profoundly persuasive in the actions of sales group's initiatives. Priyan and Uthayakumar [21] reported a probabilistic imperfect retailer-buyer combined inventory model using the deliberation of worthy investigation faults on the purchaser's and structure price as the principal asset. Joint inventory typical is extended to conclude the optimum solutions of decision variables after the seller towards the retailer in a single production run. Sundara Rajan and Uthayakumar [16] derived the typical economic order size of inventory, which was measured subject to persuasive exertions. They found a unique optimal replenishment schedule.

The residual parts of the paper were assembled as plans. In Section 2, preliminaries are presented. Section 3 hosts notations and assumptions about the typical extension of inventory manufacture. In Section 4, the crisp production-inventory system remains framed. Section 5 examines the fuzzy inventory production system. Section 6 demonstrates the crisp and fuzzy production-inventory system through a numerical sample. Also in this section, a sensitiveness test is delivered. In Section 7, comparative study is offered. Finally, it affords a conclusion and future research directions.

2. Preliminaries

The fuzzy-integrated expected total cost in a two-level production inventory system constructed on the graded mean integration technique and preliminary results are given below:

2.1 Defuzzification for Pentagonal Fuzzy Number (Harish Nagar and Priyanka Surana [8])

In Figure 1, the graded mean integration technique for $\tilde{\alpha}$ is defined by $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ as a pentagonal fuzzy number. Then the defuzzification

$$P(\tilde{\alpha}) = \frac{1}{2}\frac{\int_0^1 \frac{h}{2}[\alpha_1 + \alpha_2 + (\alpha_3 - \alpha_1)h + \alpha_4 + \alpha_5 - (\alpha_5 - \alpha_3)h]dh}{\int_0^1 hdh} \tag{1}$$

$$P(\tilde{\alpha}) = \frac{1}{12}(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + \alpha_5). \tag{2}$$

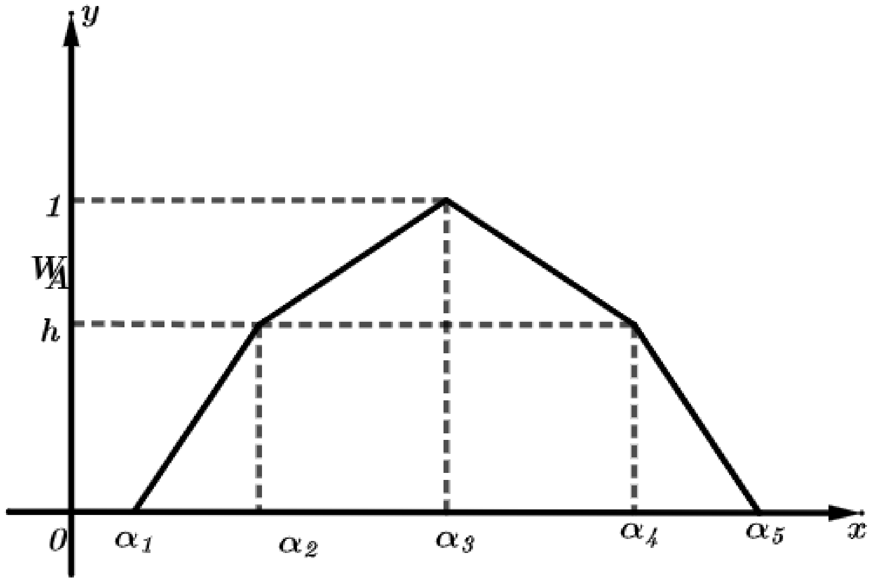


Figure 1: Pentagonal fuzzy number

2.2. Arithmetic operations on pentagonal fuzzy numbers (Kamble [10])

We define some fuzzy arithmetic operations for the pentagonal fuzzy numbers as follows.

Let $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4,\ \beta_5)$ be two pentagonal fuzzy numbers, then

1. Sum of $\tilde{\alpha}$ and $\tilde{\beta}$ indicated in $\tilde{\alpha} \oplus \tilde{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \alpha_4 + \beta_4, \alpha_5 + \beta_5)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are any real numbers.
2. Product of $\tilde{\alpha}$ and $\tilde{\beta}$ indicated in $\tilde{\alpha} \otimes \tilde{\beta} = (\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \alpha_4\beta_4, \alpha_5\beta_5)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ remain any positive real numbers.
3. If $-\tilde{\beta} = (-\beta_5, -\beta_4, -\beta_3, -\beta_2, -\beta_1)$, then the subtraction of $\tilde{\beta}$ from $\tilde{\alpha}$ is $\tilde{\alpha}\,!\,\tilde{\beta} = (\alpha_1 - \beta_5, \alpha_2 - \beta_4, \alpha_3 - \beta_3, \alpha_4 - \beta_2, \alpha_5 - \beta_1)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are any real numbers.

4. $\frac{1}{\tilde{\beta}} = \tilde{\beta}^{-1} = \left(\frac{1}{\beta_5}, \frac{1}{\beta_4}, \frac{1}{\beta_3}, \frac{1}{\beta_2}, \frac{1}{\beta_1}\right)$, where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are entirely non-zero positive real numbers, then division of $\tilde{\alpha}$ and $\tilde{\beta}$ is $\tilde{\alpha} \% \tilde{\beta} = \left(\frac{\alpha_1}{\beta_5}, \frac{\alpha_2}{\beta_4}, \frac{\alpha_3}{\beta_3}, \frac{\alpha_4}{\beta_2}, \frac{\alpha_5}{\beta_1}\right)$.

5. Any real number g, $g \otimes \tilde{\alpha} = \begin{cases} (g\alpha_1, g\alpha_2, g\alpha_3, g\alpha_4, g\alpha_5), g \geq 0 \\ (g\alpha_5, g\alpha_4, g\alpha_3, g\alpha_2, g\alpha_1), g < 0. \end{cases}$

2.3. The Kuhn–Tucker Conditions

Theorem 1. Adopt an objective function $\Phi(\Upsilon)$ and the constraints $\mathrm{E}_1(\Upsilon), \mathrm{E}_2(\Upsilon), ..., \mathrm{E}_t(\Upsilon)$, are differentiable satisfying definite regularity situations. Then $\Upsilon^* = (\gamma_1^*, \gamma_2^*, ..., \gamma_s^*)$ is the optimum determination for the nonlinear encoding problem only if there exists t numbers $\lambda_1, \lambda_2, ..., \lambda_t$, such that entirely the following settings are satisfied: $\frac{\partial \Phi}{\partial \gamma_j} - \sum_{i=1}^{t} \lambda_i \frac{\partial \mathrm{E}_i}{\partial \gamma_j} \geq 0$ at $\gamma = \gamma^*$ for $j = 1, 2, ..., s.$

1. $\gamma_j^* \frac{\partial \Phi}{\partial \gamma_j} - \sum_{i=1}^{r} \lambda_i \frac{\partial \mathrm{E}_i}{\partial \gamma_j} = 0$ at $\gamma = \gamma^*$ for $j = 1, 2, ..., s.$
2. $\mathrm{E}_i(\gamma)^* - \beta_i \leq 0$ for $i = 1, 2, ..., t.$
3. $\lambda_i(\mathrm{E}_i(\gamma)^* - \beta_i) = 0$ for $j = 1, 2, ..., s.$
4. $\gamma_j^* \geq 0$ for $i = 1, 2, ..., t.$
   $\lambda_i \geq 0$

3. Notations and assumptions

We adopt the following notations and assumptions to develop the mathematical model:

3.1 Notations

y	Production quantity,
φ	Trade group's exertions (1-point, 2- points, etc.),
K	Shortage level,
$D(\varphi)$	Demand proportion of the goods,
D_f	The principal portion of the demand proportion and that liberated of the trade group's exertions (φ),
D_s	A scale factor of the secondary part of the demand, which differs with the

	trade group's exertions (φ),
n	Price per unit effort of the trade group's exertions,
R	Production proportion,
M	Manufacture's setup cost,
O	Vendor's ordering cost,
H_f	Manufactured item's holding cost per unit/unit period,
C_1	Retrieve cost containing removal cost/unit,
C_3	Shortage cost / unit /unit period,
H_r	Raw material's holding cost /unit / unit period,
r	Flexibility factor,
s	Checking ratio / unit,
C_2	Checking cost / unit,
θ	Distribution parameter $(0 \le \theta \le 1)$ cost of trade group's exertions,
ζ	Probability mass function $f(\zeta)$ within faulty products proportionate probability,
δ	Probability mass function $f(\zeta)$ within faultless products $(\delta = 1 - \zeta)$ proportionate probability,
μ_0	Time at adjusting the shortages,
μ_1	Time span for satisfies the demand,
$\Psi(y, K, \varphi)$	Crisp integrated expected total cost per unit quantity,
$\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi)$	Fuzzy integrated expected total cost per unit quantity.

3.2 Assumptions

1. A single supply chain between manufacturer and vendor is deliberated in addition to denoting a combined system, which produces optimal policies integrated to attain a minimum cost for the supply chain.
2. The manufacturer's shortages are unacceptable, but the vendor's shortages are acceptable and these are entirely backlogged.

4. Mathematical formulation

In the manufacturing inventory system, the rate of demand for the final consumers is specified through the succeeding manifestation $D(\varphi) = D_f + D_s(1 - \frac{1}{1+\varphi})$ is associated with Cardenas-Barron and Sana [12].

4.1. Individual total cost for manufacturer

The faultless-condition basic resources inventory cost is $\frac{H_r}{2}\delta y T_1 = \frac{H_r}{2R}\delta^2 y^2$. At the same time the fault-condition basic resources inventory cost is $H_r \zeta y \frac{y}{s} = \frac{y^2 H_r \zeta}{s}$. Furthermore, the completed holding goods inventory cost $\frac{H_f}{2R}\delta^2 y^2$ is obtainable. The fault-condition basic resources removal cost for the manufacturer is $\zeta y C_1$. The checking cost is yC_2. The manufacturer's setup cost is M. The trade group's exertions distribution cost is $n\theta\varphi^r$. The manufacturer's inventory outline is displayed in Figure 2. Hence, the unit quantity's expected total cost, which is the summation of the above-mentioned costs for the manufacturer, is considered as:

$$\Psi_M = \frac{M}{E[\delta]y} + H_r\left(\frac{yE[\delta]}{2R} + \frac{yE[\zeta]}{sE[\delta]}\right) + \frac{H_f yE[\delta]}{2R} + \frac{E[\zeta]C_1}{E[\delta]} + \frac{C_2}{E[\delta]} + \frac{n\theta\varphi^r}{E[\delta]y}. \qquad (3)$$

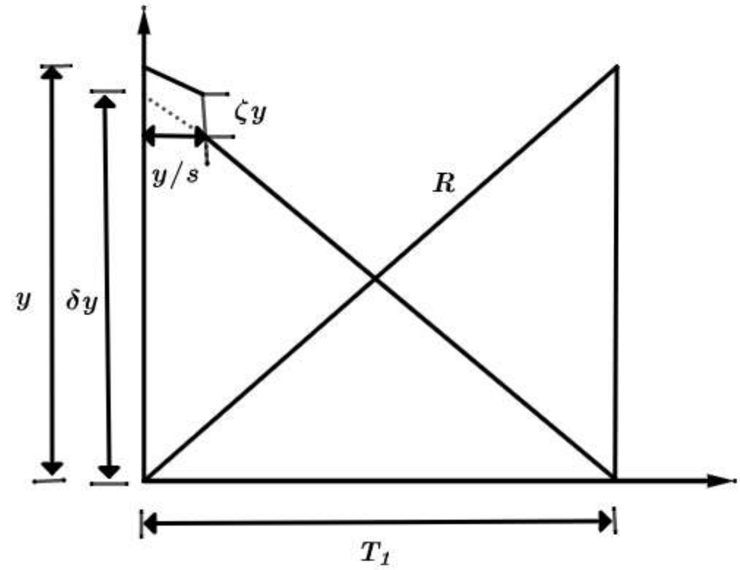


Figure 2: The manufacturer's inventory position versus time.

4.2. Individual total cost for vendor

The holding cost of the vendor is $\frac{1}{2}H_f\{\delta y - K\}\mu_1 = \frac{1}{2D}H_f\{\delta y - K\}^2$, the shortage cost is $\frac{1}{2}KC_3\mu_0 = \frac{1}{2D}C_3K^2$, the vendor's ordering cost is O, and then trade group's exertions distribution cost is $(1-\theta)n\varphi^r$. In Figure 3, the outline of vendor inventory is exposed. Henceforth, we define the total expected cost per unit quantity, which includes ordering, holding, sharing initiatives with the vendor and shortage costs is specified by

$$\Psi_V = \frac{O}{E[\delta]y} + H_f \frac{\{E[\delta]y - K\}^2}{2DE[\delta]y} + \frac{C_3K^2}{2DE[\delta]y} + \frac{n(1-\theta)\varphi^r}{E[\delta]y}. \tag{4}$$

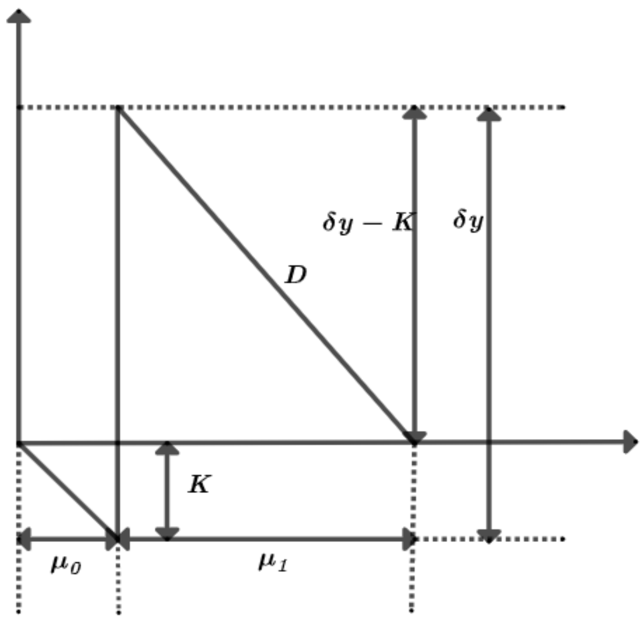


Figure 3: The vendor's inventory position versus time.

4.3. Integrated total cost of the manufacture and vendor

Affording into the assumptions, then the equations (3) to (4) defined above, (Priyan et.al. [22]) the system's integrated expected total cost represented in $\Psi(y,K,\varphi)$ comprise total expected cost of manufacturer Ψ_M and expected total cost of vendor Ψ_V. The crisp system's integrated expected total cost perhaps mathematically expressed as

$$\Psi(y,K,\varphi) = \Psi_M + \Psi_V = \frac{M}{y\delta} + H_r\left(\frac{\delta y}{2R} + \frac{\zeta y}{s\delta}\right) + \frac{H_f\delta y}{2R} + \frac{\zeta C_1}{\delta} + \frac{C_2}{\delta} + \frac{n\theta\varphi^r}{y\delta} + \frac{O}{y\delta} + H_f\frac{\{\delta y - K\}^2}{2D\delta y} + \frac{C_3K^2}{2Dy\delta} + \frac{(1-\theta)n\varphi^r}{y\delta},$$

$$\Psi(y,K,\varphi) = \frac{M+O}{\delta y} + \frac{(H_r+H_f)\delta y}{2R} + \frac{H_r\zeta y}{s\delta} + \frac{(\zeta C_1 + C_2)}{\delta} + \frac{H_f}{2D}\left[\delta y + \frac{K^2}{\delta y} - 2K\right] + \frac{C_3K^2}{2D\delta y} + \frac{n\varphi^r}{\delta y}. \tag{5}$$

Here, the integrated expected total cost $\Psi(y,K,\varphi)$ in which the terms δ and ζ in equation (5) are reformed by their expected values of $E[\delta]$ and $E[\zeta]$ correspondingly.

For the crisp case given in equation (5), the system's integrated total cost is $\Psi(y,K,\varphi)$ if entire basic resources of order lot sizes are faultless $\zeta = 0.$

$$\Psi(y,K,\varphi)=\frac{(M+O)}{y}+\frac{(H_r+H_f)y}{2R}+\frac{H_r y}{s}+C_2+\frac{H_f}{2D}(y+\frac{K^2}{y}-2K)+\frac{C_3K^2}{2D}+\frac{n\varphi^r}{y}. \quad (6)$$

4.4. Inventory model in crisp case

In the first case, we handle the production inventory system with persuasive exertion-reliant demand in a crisp manner.

4.4.1. Solution procedure

A synchronization problem involves minimizing equation (5) and determining optimal values for production lot-size $y,$ backlog K and sales team initiatives $\varphi.$

Property: 1 As a static y and $\varphi,$ $\Psi(y,K,\varphi)$ is convex in $K.$

Proof. Partial derivatives of $\Psi(y,K,\varphi)$ with respect to $K,$ produce

$$\frac{\partial\Psi(y,K,\varphi)}{\partial K}=-\frac{H_f}{D}+\frac{K(H_f+C_3)}{D\delta y}, \quad (7)$$

and

$$\frac{\partial^2\Psi(y,K,\varphi)}{\partial K^2}=\frac{(H_f+C_3)}{D\delta y}>0. \quad (8)$$

And so, for static y and $\varphi,$ $\Psi(y,K,\varphi)$ is convex in $K.$

For stable values of K and $y,$ $\Psi(y,K,\varphi)$ is a convex function in $\varphi.$ In this way, the search for optimal sales initiatives is reduced to an inspect for a locally optimal solution. Besides, by Property 1 $\Psi(y,K,\varphi)$ is subsequently convex in K for specified values φ with y. The distinct optimum based on K (indicated as K^*) feasible to acquire in equation (7) to zero as

$$K^*=(\frac{H_f}{H_f+C_3})\delta y^*. \quad (9)$$

Property: 2 As a static $\varphi,$ $\Psi(y,K,\varphi)$ is convex in $y.$

Proof. Partial derivatives of $\Psi(y,K,\varphi)$ with respect to $y,$ produce

$$\frac{\partial\Psi(y,K,\varphi)}{\partial y}=-\frac{(M+O)}{\delta y^2}+\frac{(H_f+H_r)\delta}{2R}+\frac{H_r\zeta}{s\delta}+\frac{H_f}{2D}(\delta-\frac{K^2}{\delta y^2})-\frac{C_3K^2}{2D\delta y^2}-\frac{n\varphi^r}{\delta y^2}. \quad (10)$$

Substitute equation (9) in equation (10) then we get,

$$\frac{\partial\Psi(y,K,\varphi)}{\partial y} = -\frac{(M+O)}{\delta y^2} + \frac{(H_f+H_r)\delta}{2R} + \frac{H_r\zeta}{s\delta} + \frac{H_f\delta}{2D} - \frac{H_f^{\,2}\delta}{2D(H_f+C_3)} - \frac{n\varphi^r}{\delta y^2} \tag{11}$$

and

$$\frac{\partial^2\Psi(y,K,\varphi)}{\partial y^2} = \frac{2(M+O+n\varphi^r)}{\delta y^3} > 0. \tag{12}$$

Therefore, for static φ, $\Psi(y,K,\varphi)$ is convex in y.

In addition, by Property 2 $\Psi(y,K,\varphi)$ is convex in y as specified value φ. The distinct optimal production lot size value of y (indicated to y^*) can be derived through equation (11) to zero as

$$y^* = \left\{ \frac{2(M+O+n\varphi^r)}{\delta^2\left(\frac{(H_r+H_f)}{R} + \frac{2H_r\zeta}{s\delta^2} + \frac{H_f}{D}(1-(\frac{H_f}{H_f+C_3}))\right)} \right\}^{1/2}. \tag{13}$$

The subsequent algorithm is utilized into determine the optimum results for y, K and φ as the projected crisp case.

Algorithm: 1

Step 1. Fix the trade group's exertions $\varphi = 1$.

Step 2. Compute $y_{(\varphi)}$ by using equation (13).

Step 3. Calculate $K_{(\varphi)}$ beginning at equation (9) through $y_{(\varphi)}$.

Step 4. Find the consequent $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi)$ by equation (5).

Step 5. Fix $\varphi = \varphi + 1$, recurrence Steps 2–4 to acquire $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi)$.

Step 6. If $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi) \geq \Psi(y_{(\varphi-1)}, K_{(\varphi-1)}, \varphi-1)$, then continue to Step 7 else proceed to Step 5.

Step 7. Fix $(y_{(\varphi-1)}, K_{(\varphi-1)}, \varphi-1) = (y^*, K^*, \varphi^*)$ then $\Psi(y^*, K^*, \varphi^*)$ is minimum integrated expected total cost, (y^*, K^*, φ^*) is optimum result of crisp inventory model.

5. Fuzzy model

In the second case, we handle the production inventory system with persuasive exertion-reliant demand in a fuzzy manner.

5.1 Solution procedure

The fuzzy integrated inventory model is obtainable through the equation (5); i.e., by fuzzifying the decision variables (y, K, φ) and the input parameters $(C_1, C_2, H_r, H_f, s, O, M, n, R)$.

The fuzzy form of integrated expected total cost in equation (5) is given as

$$\begin{aligned}\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi) = {} & ((\tilde{M} \oplus \tilde{O}) \% (\delta \otimes \tilde{y})) \oplus (((\tilde{H}_r \oplus \tilde{H}_f) \otimes (\delta \otimes \tilde{y})) \% (2 \otimes \tilde{R})) \oplus ((\tilde{H}_r \otimes \zeta \otimes \tilde{y}) \% \\ & (\tilde{s} \otimes \delta)) \oplus (((\zeta \otimes \tilde{C}_1) \oplus \tilde{C}_2) \% \delta) \oplus (((\tilde{H}_f) \% (2 \otimes D)) \otimes [(\tilde{y} \otimes \delta) \oplus (\tilde{K}^2 \% \delta \otimes \tilde{y}) \\ & ! \ (\tilde{K} \otimes 2)]) \oplus ((C_3 \otimes \tilde{K}^2) \% (2 \otimes D \otimes \delta \otimes \tilde{y})) \oplus ((\tilde{n} \otimes \varphi^r) \% (\delta \otimes \tilde{y})),\end{aligned} \tag{14}$$

where $\%, \oplus, !$, and $\otimes$ are the fuzzy arithmetical operatives under function principle. Here, we adopt that each input parameter $\tilde{R} = (R_1, R_2, R_3, R_4, R_5)$. $\tilde{C}_1 = (C_{11}, C_{12}, C_{13}, C_{14}, C_{15})$, $\tilde{O} = (O_1, O_2, O_3, O_4, O_5)$, $\tilde{H}_r = (H_{r1}, H_{r2}, H_{r3}, H_{r4}, H_{r5})$, $\tilde{H}_f = (H_{f1}, H_{f2}, H_{f3}, H_{f4}, H_{f5})$, $\tilde{s} = (s_1, s_2, s_3, s_4, s_5)$, $\tilde{C}_2 = (C_{21}, C_{22}, C_{23}, C_{24}, C_{25})$, $\tilde{M} = (M_1, M_2, M_3, M_4, M_5)$, $\tilde{n} = (n_1, n_2, n_3, n_4, n_5)$, are positive pentagonal fuzzy number including the five modules. Furthermore we adopt the decision variables which are fuzzified affording to the pentagonal rule as: $\tilde{K} = (K_1, K_2, K_3, K_4, K_5)$, $\tilde{y} = (y_1, y_2, y_3, y_4, y_5)$.

We initiate through, fuzzy integrated expected total cost $\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi)$ which is given by equation (14). Then

$$\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi) = \left(\begin{aligned} & \frac{M_1 + O_1}{\delta y_5} + \frac{(H_{r1} + H_{f1})\delta y_1}{2R_5} + \frac{H_{r1}\zeta y_1}{s_5 \delta} + \frac{(\zeta C_{11} + C_{21})}{\delta} + \frac{H_{f1}}{2D}[\delta y_1 + \frac{K_1^{\,2}}{\delta y_5} - 2K_5] + \frac{C_3 K_1^{\,2}}{2D\delta y_5} + \frac{n_1 \varphi^r}{\delta y_5}, \\ & \frac{M_2 + O_2}{\delta y_4} + \frac{(H_{r2} + H_{f2})\delta y_2}{2R_4} + \frac{H_{r2}\zeta y_2}{s_4 \delta} + \frac{(\zeta C_{12} + C_{22})}{\delta} + \frac{H_{f2}}{2D}[\delta y_2 + \frac{K_2^{\,2}}{\delta y_4} - 2K_4] + \frac{C_3 K_2^{\,2}}{2D\delta y_4} + \frac{n_2 \varphi^r}{\delta y_4}, \\ & \frac{M_3 + O_3}{\delta y_3} + \frac{(H_{r3} + H_{f3})\delta y_3}{2R_3} + \frac{H_{r3}\zeta y_3}{s_3 \delta} + \frac{(\zeta C_{13} + C_{23})}{\delta} + \frac{H_{f3}}{2D}[\delta y_3 + \frac{K_3^{\,2}}{\delta y_3} - 2K_3] + \frac{C_3 K_3^{\,2}}{2D\delta y_3} + \frac{n_3 \varphi^r}{\delta y_3}, \\ & \frac{M_4 + O_4}{\delta y_2} + \frac{(H_{r4} + H_{f4})\delta y_4}{2R_2} + \frac{H_{r4}\zeta y_4}{s_2 \delta} + \frac{(\zeta C_{14} + C_{24})}{\delta} + \frac{H_{f4}}{2D}[\delta y_4 + \frac{K_4^{\,2}}{\delta y_2} - 2K_2] + \frac{C_3 K_4^{\,2}}{2D\delta y_2} + \frac{n_4 \varphi^r}{\delta y_2}, \\ & \frac{M_5 + O_5}{\delta y_1} + \frac{(H_{r5} + H_{f5})\delta y_5}{2R_1} + \frac{H_{r5}\zeta y_5}{s_1 \delta} + \frac{(\zeta C_{15} + C_{25})}{\delta} + \frac{H_{f5}}{2D}[\delta y_5 + \frac{K_5^{\,2}}{\delta y_1} - 2K_1] + \frac{C_3 K_5^{\,2}}{2D\delta y_1} + \frac{n_5 \varphi^r}{\delta y_1} \end{aligned} \right). \tag{15}$$

We defuzzify $\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi)$ by equation (2) then get through the graded mean integration technique of $\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi)$

$$
\begin{aligned}
P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)) = \Bigg(& \frac{1}{12}\left(\frac{M_1+O_1}{\delta y_5} + \frac{(H_{r1}+H_{f1})\delta y_1}{2R_5} + \frac{H_{r1}\zeta y_1}{s_5\delta} + \frac{(\zeta C_{11}+C_{21})}{\delta} + \frac{H_{f1}}{2D}[\delta y_1 + \frac{K_1^{\,2}}{\delta y_5} - 2K_5] + \frac{C_3 K_1^{\,2}}{2D\delta y_5} + \frac{n_1\varphi^r}{\delta y_5} \right) \\
& + \frac{3}{12}\left(\frac{M_2+O_2}{\delta y_4} + \frac{(H_{r2}+H_{f2})\delta y_2}{2R_4} + \frac{H_{r2}\zeta y_2}{s_4\delta} + \frac{(\zeta C_{12}+C_{22})}{\delta} + \frac{H_{f2}}{2D}[\delta y_2 + \frac{K_2^{\,2}}{\delta y_4} - 2K_4] + \frac{C_3 K_2^{\,2}}{2D\delta y_4} + \frac{n_2\varphi^r}{\delta y_4} \right) \\
& + \frac{4}{12}\left(\frac{M_3+O_3}{\delta y_3} + \frac{(H_{r3}+H_{f3})\delta y_3}{2R_3} + \frac{H_{r3}\zeta y_3}{s_3\delta} + \frac{(\zeta C_{13}+C_{23})}{\delta} + \frac{H_{f3}}{2D}[\delta y_3 + \frac{K_3^{\,2}}{\delta y_3} - 2K_3] + \frac{C_3 K_3^{\,2}}{2D\delta y_3} + \frac{n_3\varphi^r}{\delta y_3} \right) \\
& + \frac{3}{12}\left(\frac{M_4+O_4}{\delta y_2} + \frac{(H_{r4}+H_{f4})\delta y_4}{2R_2} + \frac{H_{r4}\zeta y_4}{s_2\delta} + \frac{(\zeta C_{14}+C_{24})}{\delta} + \frac{H_{f4}}{2D}[\delta y_4 + \frac{K_4^{\,2}}{\delta y_2} - 2K_2] + \frac{C_3 K_4^{\,2}}{2D\delta y_2} + \frac{n_4\varphi^r}{\delta y_2} \right) \\
& + \frac{1}{12}\left(\frac{M_5+O_5}{\delta y_1} + \frac{(H_{r5}+H_{f5})\delta y_5}{2R_1} + \frac{H_{r5}\zeta y_5}{s_1\delta} + \frac{(\zeta C_{15}+C_{25})}{\delta} + \frac{H_{f5}}{2D}[\delta y_5 + \frac{K_5^{\,2}}{\delta y_1} - 2K_1] + \frac{C_3 K_5^{\,2}}{2D\delta y_1} + \frac{n_5\varphi^r}{\delta y_1} \right)\Bigg),
\end{aligned}
\tag{16}
$$

where $0 < K_1 \le K_2 \le K_3 \le K_4 \le K_5$ and $0 < y_1 \le y_2 \le y_3 \le y_4 \le y_5$. We esteem $P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi))$ in this way determine for the integrated expected total cost per unit as the fuzzy sense. We can exchange inequality conditions $0 \le K_1 \le K_2 \le K_3 \le K_4 \le K_5$, $0 \le y_1 \le y_2 \le y_3 \le y_4 \le y_5$ without changing the meaning of equation (16). Thus, the optimal solution of given in equation (16), subject to the succeeding dissimilarity constraints: $y_4 - y_5 \le 0,\ y_3 - y_4 \le 0,\ y_2 - y_3 \le 0,\ y_1 - y_2 \le 0,\ -y_1 < 0$ and $K_4 - K_5 \le 0$, $K_3 - K_4 \le 0,\ K_2 - K_3 \le 0,\ K_1 - K_2 \le 0,\ -K_1 < 0.$

An optimum production, shortage level of $P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)$ is found by applying the Kuhn–Tucker conditions subject to ten inequalities as imposed conditions. Kuhn–Tucker conditions on Theorem 1 as follows $\lambda \ge 0$,

$$\nabla P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi) - \lambda \nabla \mathrm{E}(\tilde{y},\tilde{K},\varphi) = 0, \lambda \mathrm{E}(\tilde{y},\tilde{K},\varphi) = 0, \mathrm{E}(\tilde{y},\tilde{K},\varphi) \le 0.$$

These conditions simplify to the following $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10} \ge 0$,

$$
\begin{aligned}
& \nabla P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi) - \lambda_1(K_1-K_2) - \lambda_2(K_2-K_3) - \lambda_3(K_3-K_4) - \lambda_4(K_4-K_5) - \lambda_5(-K_1) \\
& -\lambda_6(y_1-y_2) - \lambda_7(y_2-y_3) - \lambda_8(y_3-y_4) - \lambda_9(y_4-y_5) - \lambda_{10}(-y_1) = 0,
\end{aligned}
\tag{17}
$$

$$\frac{1}{12}\left[\frac{H_{f1}K_1}{D\delta y_5} + \frac{C_3 K_1}{D\delta y_5} - \frac{H_{f5}}{D}\right] - \lambda_1 + \lambda_5 = 0, \tag{18}$$

$$\frac{3}{12}\left[\frac{H_{f2}K_2}{D\delta y_4} + \frac{C_3 K_2}{D\delta y_4} - \frac{H_{f4}}{D}\right] - \lambda_2 + \lambda_1 = 0, \tag{19}$$

$$\frac{4}{12}\left[\frac{H_{f3}K_3}{D\delta y_3} + \frac{C_3 K_3}{D\delta y_3} - \frac{H_{f3}}{D}\right] - \lambda_3 + \lambda_2 = 0, \tag{20}$$

$$\frac{3}{12}\left[\frac{H_{f4}K_4}{D\delta y_2} + \frac{C_3 K_4}{D\delta y_2} - \frac{H_{f2}}{D}\right] - \lambda_4 + \lambda_3 = 0, \tag{21}$$

$$\frac{1}{12}\left[\frac{H_{f5}K_5}{D\delta y_1} + \frac{C_3 K_5}{D\delta y_1} - \frac{H_{f1}}{D}\right] + \lambda_4 = 0, \tag{22}$$

$$\frac{1}{12}\left[-\frac{(M_5+O_5)}{\delta y_1^{\,2}}+\frac{(H_{f1}+H_{r1})\delta}{2R_5}+\frac{H_{r1}\zeta}{s_5\delta}+\frac{H_{f1}\delta}{2D}-\frac{H_{f5}K_5^{\,2}}{2D\delta y_1^{\,2}}-\frac{C_3K_5^{\,2}}{2D\delta y_1^{\,2}}-\frac{n_5\varphi^r}{\delta y_1^{\,2}}\right]-\lambda_6+\lambda_{10}=0 \quad (23)$$

$$\frac{3}{12}\left[-\frac{(M_4+O_4)}{\delta y_2^{\,2}}+\frac{(H_{f2}+H_{r2})\delta}{2R_4}+\frac{H_{r2}\zeta}{s_4\delta}+\frac{H_{f2}\delta}{2D}-\frac{H_{f4}K_4^{\,2}}{2D\delta y_2^{\,2}}-\frac{C_3K_4^{\,2}}{2D\delta y_2^{\,2}}-\frac{n_4\varphi^r}{\delta y_2^{\,2}}\right]-\lambda_7+\lambda_6=0 \quad (24)$$

$$\frac{4}{12}\left[-\frac{(M_3+O_3)}{\delta y_3^{\,2}}+\frac{(H_{f3}+H_{r3})\delta}{2R_3}+\frac{H_{r3}\zeta}{s_3\delta}+\frac{H_{f3}\delta}{2D}-\frac{H_{f3}K_3^{\,2}}{2D\delta y_3^{\,2}}-\frac{C_3K_3^{\,2}}{2D\delta y_3^{\,2}}-\frac{n_3\varphi^r}{\delta y_3^{\,2}}\right]-\lambda_8+\lambda_7=0 \quad (25)$$

$$\frac{3}{12}\left[-\frac{(M_2+O_2)}{\delta y_4^{\,2}}+\frac{(H_{f4}+H_{r4})\delta}{2R_2}+\frac{H_{r4}\zeta}{s_2\delta}+\frac{H_{f4}\delta}{2D}-\frac{H_{f2}K_2^{\,2}}{2D\delta y_4^{\,2}}-\frac{C_3K_2^{\,2}}{2D\delta y_4^{\,2}}-\frac{n_2\varphi^r}{\delta y_4^{\,2}}\right]-\lambda_9+\lambda_8=0 \quad (26)$$

$$\frac{1}{12}\left[-\frac{(M_1+O_1)}{\delta y_5^{\,2}}+\frac{(H_{f2}+H_{r2})\delta}{2R_1}+\frac{H_{r5}\zeta}{s_1\delta}+\frac{H_{f5}\delta}{2D}-\frac{H_{f1}K_1^{\,2}}{2D\delta y_5^{\,2}}-\frac{C_3K_1^{\,2}}{2D\delta y_5^{\,2}}-\frac{n_1\varphi^r}{\delta y_5^{\,2}}\right]+\lambda_9=0 \quad (27)$$

$K_i-K_{i+1}\le 0,\quad i=1,2,3,4,$ (28)

$K_1>0,$ (29)

$y_i-y_{i+1}\le 0,\quad i=1,2,3,4,$ (30)

$y_1>0,$ (31)

$\lambda_i(K_i-K_{i+1})=0,\quad i=1,2,3,4,$ (32)

$\lambda_5(K_1)=0,$ (33)

$\lambda_{5+i}(y_i-y_{i+1})=0,\quad i=1,2,3,4,$ (34)

$\lambda_{10}(y_1)=0,$ (35)

$K_i\ge 0,\ y_i\ge 0,\ \ i=1,2,3,4,5$ and $\lambda_j\ge 0,\ \ j=1,2,.....,10.$ (36)

Because $K_1>0$, and $\lambda_5K_1=0$, then $\lambda_5=0$. If $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$, then $K_5<K_4<K_3<K_2<K_1$, it does not satisfy the constraints $K_5\ge K_4\ge K_3\ge K_2\ge K_1>0$. Therefore, $K_5=K_4, K_4=K_3, K_3=K_2, K_2=K_1$, that is $K_5=K_4=K_3=K_2=K_1=\tilde{K}^*$. Since $y_1>0$, and $\lambda_{10}y_1=0$, then $\lambda_{10}=0$. If $\lambda_6=\lambda_7=\lambda_8=\lambda_9=0$, then $y_5<y_4<y_3<y_2<y_1$, that cannot satisfy the constraints $y_5\ge y_4\ge y_3\ge y_2\ge y_1>0$. Hence, $y_5=y_4, y_4=y_3, y_3=y_2, y_2=y_1$, it is $y_5=y_4=y_3=y_2=y_1=\tilde{y}^*$. Henceforth, from equations (18)–(27), we determine the fuzzy optimum production quantity $\tilde{y}^*$ and shortage level quantity $\tilde{K}^*$ as

$$\tilde{y}^*=\left\{\frac{P+3[S]+4[T]+3[U]+V}{\delta^2(E+3F+4G+3H+I)}\right\}^{1/2}, \quad (37)$$

$$\tilde{K}^*=\left(\frac{H_{f5}\delta y_5+3H_{f4}\delta y_4+4H_{f3}\delta y_3+3H_{f2}\delta y_2+H_{f1}\delta y_1}{(H_{f1}+C_3)+3(H_{f2}+C_3)+4(H_{f3}+C_3)+3(H_{f4}+C_3)+(H_{f5}+C_3)}\right). \quad (38)$$

Where, $P = 2(M_5 + O_5 + n_5\varphi^r)$, $S = 2(M_4 + O_4 + n_4\varphi^r)$, $T = 2(M_3 + O_3 + n_3\varphi^r)$, $U = 2(M_2 + O_2 + n_2\varphi^r), V = 2(M_1 + O_1 + n_1\varphi^r),$ $E = \frac{(H_{r1} + H_{f1})}{R_5} + \frac{2H_{r1}\zeta}{s_5\delta^2} + \frac{H_{f1}}{D}(1 - (\frac{H_{f1}}{H_{f5} + C_3})),$

$F = \frac{(H_{r2} + H_{f2})}{R_4} + \frac{2H_{r2}\zeta}{s_4\delta^2} + \frac{H_{f2}}{D}(1 - (\frac{H_{f2}}{H_{f4} + C_3})),$ $G = \frac{(H_{r3} + H_{f3})}{R_3} + \frac{2H_{r3}\zeta}{s_3\delta^2} + \frac{H_{f3}}{D}(1 - (\frac{H_{f3}}{H_{f3} + C_3})),$

$H = \frac{(H_{r4} + H_{f4})}{R_2} + \frac{2H_{r4}\zeta}{s_2\delta^2} + \frac{H_{f4}}{D}(1 - (\frac{H_{f4}}{H_{f2} + C_3})),$ $I = \frac{(H_{r5} + H_{f5})}{R_1} + \frac{2H_{r5}\zeta}{s_1\delta^2} + \frac{H_{f5}}{D}(1 - (\frac{H_{f5}}{H_{f1} + C_3})).$

The optimum integrated expected total cost $P(\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi))$ is found through direct substitution of equations (37) and (38) into equation (16).

It is noted that when the parameters $(C_1, C_2, H_r, H_f, s, O, M, n, R)$ are real numbers, that is, $C_{15} = C_{14} = C_{13} = C_{12} = C_{11} = C_1$, $C_{25} = C_{24} = C_{23} = C_{22} = C_{21} = C_2$, $H_{r5} = H_{r4} = H_{r3} = H_{r2} = H_{r1} = H_r$, $H_{f5} = H_{f4} = H_{f3} = H_{f2} = H_{f1} = H_f$, $s_5 = s_4 = s_3 = s_2 = s_1 = s$, $O_5 = O_4 = O_3 = O_2 = O_1 = O$, $M_1 = M_2 = M_3 = M_4 = M_5 = M$, $n_5 = n_4 = n_3 = n_2 = n_1 = n$, $R_5 = R_4 = R_3 = R_2 = R_1 = R$. Assume that the decision variables (y, K) are real numbers, $y_5 = y_4 = y_3 = y_2 = y_1 = y$, $K_5 = K_4 = K_3 = K_2 = K_1 = K$ then equations (37) and (38) condensed as equations (9) and (13) respectively as

$$K^* = (\frac{H_f}{H_f + C_3})\delta y^*, \tag{39}$$

$$y^* = \left\{ \frac{2(M + O + n\varphi^r)}{\delta^2 \left(\frac{(H_r + H_f)}{R} + \frac{2H_r\zeta}{s\delta^2} + \frac{H_f}{D}(1 - (\frac{H_f}{H_f + C_3})) \right)} \right\}^{1/2}. \tag{40}$$

The subsequent algorithm is utilized into determine the optimum results for $\tilde{y}, \tilde{K}$ and φ as the projected fuzzy inventory case.

Algorithm: 2

Step 1. Fix the trade group's exertions $\varphi = 1$.

Step 2. Compute $\tilde{y}_{(\varphi)}$ by equation (37).

Step 3. Calculate $\tilde{K}_{(\varphi)}$ from equation (38) by $\tilde{y}_{(\varphi)}$.

Step 4. Find the consequent $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi))$ by equation (16).

Step 5. Fix $\varphi = \varphi + 1$, recurrence Steps 2–4 to acquire $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi))$.

Step 6. If $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi)) \geq P(\tilde{\Psi}(\tilde{y}_{(\varphi-1)}, \tilde{K}_{(\varphi-1)}, \varphi - 1))$, then continues Step 7 else proceed Step 5.

Step 7. Fix $(\tilde{y}_{(\varphi-1)}, \tilde{K}_{(\varphi-1)}, \varphi-1) = (\tilde{y}^*, \tilde{K}^*, \varphi^*)$ then $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*))$ is minimum integrated expected total cost, $(\tilde{y}^*, \tilde{K}^*, \varphi^*)$ is optimum result of projected fuzzy inventory model.

6. Numerical and sensitivity analysis

6.1. Numerical analysis

In the present division, a numerical sample is taken towards demonstrate that exceeding result technique. Let us deliberate the structure through the crisp data used in Priyan et.al. [22] the result towards this sample is acquired through MatLab software. The input of crisp and fuzzy parameters are proceeds: $H_r =$ \$5/unit/year, $C_1 =$ \$15/unit, $H_f =$ \$3/unit/year, $D_s = 10$units/year, $D_f = 100$units/year, $C_3 =$ \$25/unit/year, $C_2 =$ \$0.5/unit, $s = 175200$, $M =$ \$400/setup, $O =$ \$100/order, $n =$ \$1500/unit, $r = 1$, $\theta =$ 50%, $R = 150$units/year, $\tilde{H}_r =$ \$(1, 3, 5, 7, 9)/unit/year, $\tilde{H}_f =$ \$(1, 2, 3, 4, 5)/unit/year, $\tilde{C}_1 =$ \$(5, 10, 15, 20, 25)/unit, $\tilde{C}_2 =$ \$(0.1, 0.3, 0.5, 0.7, 0.9)/unit, $\tilde{s} =$ (154200, 164700, 175200, 185700, 196200), $\tilde{O} =$ \$(27, 63.5, 100, 136.5, 173)/unit/year, $\tilde{M} =$ \$(230, 315, 400, 485, 570)/setup, $\tilde{n} =$ \$(584, 1042, 1500, 1958, 2416)/unit, $\tilde{R} =$ (65, 107.5, 150, 192.5, 235)units/year, and ζ is random variable as uniformly distributed, $U_D(\alpha, \beta) =$ (0,0.04).

Here $E(\zeta) = (\alpha+\beta)/2 = 0.02$ and $E[(1-\zeta)^2] = E[\delta^2] = \dfrac{(\alpha^2+\beta^2+\alpha\beta)}{3} + 1 - \alpha - \beta =$ 0.96053. Currently using the Algorithm 2 technique produces the succeeding optimal solutions for the fuzzy model: $\tilde{y}^* = 211.0$ units, $\tilde{K}^* = 22.4$ units, $\varphi^* =$ 1 point, $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*)) =$ \$18.0/unit, and $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*)) \times \tilde{y}^* =$ \$3795.0. Similarly, we use Algorithm 1 procedure to get the following optimal solutions for crisp model: $y^* =$ 229.8 units, $K^* = 24.1$ units, $\varphi^* =$ 1 point, $\Psi(y^*, K^*, \varphi^*) =$ \$18.6/unit, and $\Psi(y^*, K^*, \varphi^*) \times y^* =$ \$4269.0.

6.2. Sensitivity analysis

We study the distinct responses of $H_r, H_f, C_1, C_2, s, O, M, n, R, C_3, D_f$, and D_s on the optimal y, K and φ to clarify the crisp and fuzzy cases and algorithms. The sensitiveness study is implemented via varying the parameters $H_r, H_f, C_1, C_2, s,$ O, M, n, R, C_3, D_f, $\tilde{H}_r, \tilde{H}_f, \tilde{C}_1, \tilde{C}_2, \tilde{s}, \tilde{O}, \tilde{M}, \tilde{n}, \tilde{R}, C_3, D_f$, and D_s by +90%, +60%, +30%, –30%, –60%, and –90%. The outcomes for crisp and fuzzy cases are offered in Tables 1 to 3 and the corresponding total cost curves are plotted in Figure 4.

Parameters	changing %	Crisp case					Fuzzy case					Fuzzy case Savings %		
		y^*	K^*	φ^*	Ψ	$\Psi \times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\Psi)$	$P(\Psi)\times\tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\Psi)\times\tilde{y}^*$
	+90	249.2	26.2	1	17.2	4284.8	225.4	24.3	1	16.7	3764.2	9.6	7.0	12.1
	+60	244.7	25.7	1	17.5	4281.1	222.1	23.9	1	17.0	3771.8	9.2	7.1	11.9
	+30	238.6	25.0	1	17.9	4276.1	217.6	23.2	1	17.4	3781.6	8.8	7.3	11.6
D_f	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	216.3	22.7	1	19.7	4257.9	200.5	21.0	1	19.0	3814.2	7.3	7.5	10.4
	−60	192.1	20.2	1	22.1	4238.2	181.2	18.6	1	21.2	3843.5	5.7	7.6	9.3
	−90	134.0	14.1	1	31.3	4190.7	130.7	13.1	1	29.7	3888.3	2.4	7.1	7.2
	+90	231.4	24.3	1	18.5	4270.2	212.2	22.5	1	17.9	3792.7	8.3	7.4	11.2
	+60	230.9	24.2	1	18.5	4269.8	211.8	22.5	1	17.9	3793.5	8.3	7.4	11.2
	+30	230.3	24.2	1	18.5	4269.4	211.4	22.4	1	17.9	3794.3	8.2	7.4	11.1
D_s	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.3	24.1	1	18.6	4268.5	210.6	22.3	1	18.0	3795.8	8.2	7.4	11.1
	−60	228.7	24.0	1	18.7	4268.1	210.2	22.2	1	18.1	3796.7	8.1	7.4	11.0
	−90	228.2	24.0	1	18.7	4267.6	209.8	22.2	1	18.1	3797.5	8.1	7.4	11.0
	+90	234.9	24.7	1	19.0	4456.8	215.7	22.8	1	18.3	3957.9	8.2	7.4	11.2
	+60	233.2	24.5	1	18.8	4394.2	214.1	22.7	1	18.2	3903.7	8.2	7.4	11.2
	+30	231.5	24.3	1	18.7	4331.6	212.6	22.5	1	18.1	3849.4	8.2	7.4	11.1
O & $\tilde{O}$	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	228.1	23.9	1	18.4	4206.3	209.4	22.2	1	17.9	3740.7	8.2	7.4	11.1
	−60	226.3	23.8	1	18.3	4143.7	207.8	22.0	1	17.7	3686.4	8.2	7.3	11.0
	−90	224.6	23.6	1	18.2	4081.0	206.2	21.9	1	17.6	3632.0	8.2	7.3	11.0
	+90	278.8	29.3	1	15.5	4308.9	261.1	27.1	1	14.9	3890.4	6.3	7.6	9.7
	+60	266.0	27.9	1	16.2	4298.5	247.8	25.8	1	15.6	3864.4	6.9	7.5	10.1
	+30	250.2	26.3	1	17.1	4285.6	231.5	24.3	1	16.6	3833.2	7.5	7.5	10.6
R & $\tilde{R}$	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	202.4	21.2	1	21.0	4246.5	184.1	19.7	1	20.4	3747.0	9.0	7.2	11.8
	−60	161.9	17.0	1	26.0	4213.5	145.7	15.8	1	25.3	3683.3	10.0	6.9	12.6
	−90	86.3	9.1	1	48.1	4151.8	76.6	8.5	1	46.8	3585.5	11.3	6.3	13.6
	+90	249.6	26.2	1	20.1	5019.8	229.2	24.3	1	19.5	4478.0	8.2	7.2	10.8
	+60	243.2	25.5	1	19.6	4769.7	223.3	23.7	1	19.0	4250.5	8.2	7.3	10.9
	+30	236.6	24.8	1	19.1	4519.4	217.2	23.0	1	18.5	4022.9	8.2	7.3	11.0
M & $\tilde{M}$	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	222.8	23.4	1	18.0	4018.4	204.6	21.7	1	17.4	3567.0	8.2	7.4	11.2
	−60	215.6	22.6	1	17.5	3767.6	197.9	20.9	1	16.9	3338.7	8.2	7.5	11.4
	−90	208.1	21.9	1	16.9	3516.6	191.1	20.2	1	16.3	3110.1	8.2	7.6	11.6

Table 1: Changing of parameters $D_f, D_s, O, \tilde{O}, R, \tilde{R}, M$ and $\tilde{M}$ on optimum result.

		Crisp case	Fuzzy case	Fuzzy case Savings

												%		
		y^*	K^*	φ^*	Ψ	$\Psi\times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\Psi)$	$P(\Psi)\times\tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\Psi)\times\tilde{y}^*$
n & $\tilde{n}$	+90	297.4	31.2	1	23.8	7079.1	273.1	28.9	1	23.0	6278.8	8.2	7.5	11.3
	+60	276.7	29.1	1	22.2	6143.9	254.1	26.9	1	21.5	5452.2	8.2	7.4	11.3
	+30	254.4	26.7	1	20.5	5207.3	233.5	24.7	1	19.8	4624.5	8.2	7.4	11.2
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	202.3	21.2	1	16.5	3328.2	185.8	19.7	1	16.0	2963.4	8.2	7.3	11.0
	−60	170.4	17.9	1	14.0	2383.9	156.5	16.6	1	13.6	2128.3	8.2	7.2	10.7
	−90	131.0	13.8	1	10.9	1433.4	120.3	12.8	1	10.7	1287.4	8.2	7.0	10.2
C_1 & $\tilde{C}_1$	+90	229.8	24.1	1	18.9	4332.3	211.0	22.4	1	18.3	3853.2	8.2	7.4	11.1
	+60	229.8	24.1	1	18.8	4311.2	211.0	22.4	1	18.2	3833.8	8.2	7.4	11.1
	+30	229.8	24.1	1	18.7	4290.1	211.0	22.4	1	18.1	3814.4	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.5	4247.9	211.0	22.4	1	17.9	3775.7	8.2	7.4	11.1
	−60	229.8	24.1	1	18.4	4226.8	211.0	22.4	1	17.8	3756.3	8.2	7.4	11.1
	−90	229.8	24.1	1	18.3	4205.6	211.0	22.4	1	17.7	3736.9	8.2	7.4	11.1
C_2 & $\tilde{C}_2$	+90	229.8	24.1	1	19.0	4374.5	211.0	22.4	1	18.4	3891.9	8.2	7.4	11.0
	+60	229.8	24.1	1	18.9	4339.3	211.0	22.4	1	18.3	3859.6	8.2	7.4	11.1
	+30	229.8	24.1	1	18.7	4304.1	211.0	22.4	1	18.1	3827.3	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.4	4233.8	211.0	22.4	1	17.8	3762.7	8.2	7.4	11.1
	−60	229.8	24.1	1	18.3	4198.6	211.0	22.4	1	17.7	3730.5	8.2	7.4	11.1
	−90	229.8	24.1	1	18.1	4163.4	211.0	22.4	1	17.5	3698.2	8.2	7.4	11.2
s & $\tilde{s}$	+90	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	+60	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	+30	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−60	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−90	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
C_3	+90	227.9	13.3	1	18.7	4267.4	209.2	12.3	1	18.1	3792.1	8.2	7.3	11.1
	+60	228.3	15.6	1	18.7	4267.7	209.6	14.5	1	18.1	3792.7	8.2	7.3	11.1
	+30	228.9	19.0	1	18.6	4268.2	210.1	17.6	1	18.1	3793.6	8.2	7.3	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	231.5	33.2	1	18.4	4270.3	212.5	30.7	1	17.9	3797.9	8.2	7.4	11.1
	−60	235.1	53.2	1	18.2	4273.3	216.1	49.2	1	17.6	3805.5	8.1	7.5	10.9
	−90	250.6	133.9	1	17.1	4285.9	233.8	123.7	1	16.5	3861.4	6.7	7.7	9.9

Table 2: Changing of parameters $n, \tilde{n}, C_1, \tilde{C}_1, C_2, \tilde{C}_2, s, \tilde{s}$ and C_3 on optimum result.

Parameters	changing %	Crisp case					Fuzzy case					Fuzzy case Savings %		
		y^*	K^*	φ^*	Ψ	$\Psi \times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\Psi)$	$P(\Psi)\times\tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\Psi)\times\tilde{y}^*$
	+90	195.6	20.5	1	21.7	4241.0	177.4	19.1	1	21.0	3728.1	9.3	6.9	12.1
	+60	205.3	21.6	1	20.7	4248.9	186.7	20.0	1	20.1	3746.3	9.0	7.1	11.8
H_r	+30	216.5	22.7	1	19.7	4258.1	197.8	21.1	1	19.1	3768.2	8.7	7.2	11.5
&	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
$\tilde{H}_r$	−30	245.9	25.8	1	17.4	4282.1	227.3	23.9	1	16.8	3829.0	7.6	7.5	10.6
	−60	266.0	27.9	1	16.2	4298.5	248.2	25.8	1	15.6	3873.4	6.7	7.7	9.9
	−90	292.0	30.7	1	14.8	4319.7	276.0	28.3	1	14.3	3934.5	5.5	7.8	8.9
	+90	189.8	34.5	1	22.3	4236.3	176.5	31.9	1	21.6	3808.0	7.0	7.8	10.1
	+60	200.4	31.6	1	21.2	4245.0	185.7	29.2	1	20.5	3804.4	7.3	7.7	10.4
H_f	+30	213.4	28.2	1	19.9	4255.5	196.9	26.1	1	19.3	3800.3	7.7	7.5	10.7
&	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
$\tilde{H}_f$	−30	251.6	19.1	1	17.0	4286.7	229.4	17.7	1	16.5	3787.5	8.8	7.1	11.6
	−60	282.3	12.7	1	15.3	4311.8	254.8	11.8	1	14.8	3774.6	9.8	6.6	12.5
	−90	330.4	3.8	1	13.2	4351.0	292.9	3.6	1	12.8	3747.1	11.3	5.5	13.9

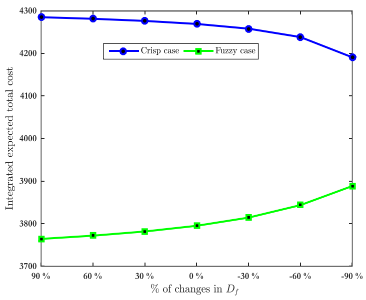
Table 3: Changing of parameters $H_r, \tilde{H}_r, H_f$ and $\tilde{H}_f$ on optimum result.

6.3. Managerial effects

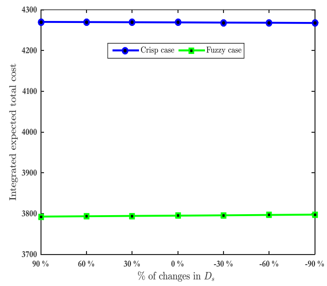
The specified certain managerial effects of the suggested case established in that arithmetical and sensitiveness study.

- According to Tables 1 and 2, together of optimal production quantities $y,$ and $\tilde{y},$ shortage level $K,$ and $\tilde{K},$ are raise as the first part of the demand rate D_f rises. The reason for that when demand increases, the production quantity and the shortage level also increase, resulting in a success situation for all parties involved. Compared to the crisp case, we find $P(\Psi)\times\tilde{y}^*$ falls with an increase in demand rate D_f. This arises owing toward the effect of shortage, holding rate, etc., on the inventory phases.
- Increasing the second portion of demand ratio D_s decreases mutually the optimal production quantity, shortage level of crisp and fuzzy case.
- As ordering costs O and $\tilde{O}$ decrease, corresponding crisp and fuzzy shortage levels $K,$ and $\tilde{K},$ as well as optimal production quantity $y,$ and $\tilde{y},$ integrated expected total costs $\Psi \times y^*$ and $P(\Psi)\times\tilde{y}^*$ decrease without affecting sales team initiatives φ^*. Similarly, it gives the same effect for the production rates R and $\tilde{R},$ the setup costs M and $\tilde{M},$ the sales teams' initiatives cost n and $\tilde{n}.$

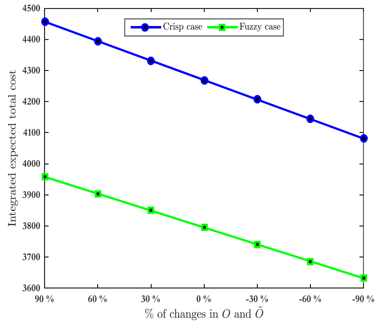
- There is a stable relationship between the crisp and fuzzy optimal production lot sizes and shortage levels when salvage costs C_1 and $\tilde{C}_1$, screening costs C_2 and $\tilde{C}_2$ decrease. Furthermore, the situation is stimulating toward annotation of unit quantity's integrated expected total cost declines though the integrated expected total cost and rises uninterrupted the trade group's exertions φ^*.
- The effects of Tables 2, 3 are clarified that the decision variables are constant while the screening rates s and $\tilde{s}$ decrease.
- Lot-sizes y and $\tilde{y}$, shortage levels K and $\tilde{K}$ increase as shortage costs C_3 and raw material's holding costs H_r and $\tilde{H}_r$ decrease. Further, a remarkable trend is observed in which the expected total costs of unit's declines while the integrated expected total costs increase without having an impact on the trade group's exertions φ^*.
- As a result of a decrease manufactured holding costs H_f and $\tilde{H}_f$ the optimal production quantity y and $\tilde{y}$ increase, shortage level K and $\tilde{K}$ decline. Moreover, it is notable these unit quantity's expected total cost decreases whereas the integrated expected total cost rises in crisp case and decreases in fuzzy case without affecting the trade group's exertions φ^*.
- Based on the fuzzy model, we observe shortage level, optimum production quantity, minimum integrated expected total cost savings, ranging from 5.5% to 7.8%, 2.4% to 11.3%, and 7.2% to 13.2%, respectively.
- Additionally, it is shown that the optimal production quantity, shortage level, and minimum integrated expected total cost of the fuzzy model are significantly different from their crisp model. The graphical demonstration of the sensitiveness study is exposed in Figure 4.
- Also, the minimum integrated expected total cost of the fuzzy model is highly significant from the results of the crisp model.



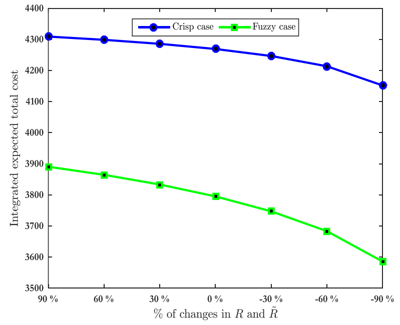
(a) Sensitivity analysis for D_f.



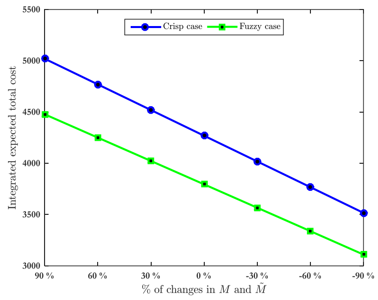
(b) Sensitivity analysis for D_s.



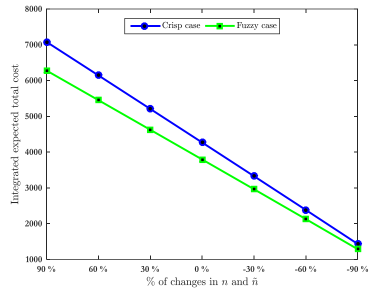
(c) Sensitivity analysis for O and $\tilde{O}$.



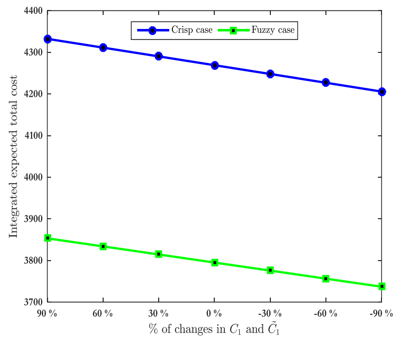
(d) Sensitivity analysis for R and $\tilde{R}$.



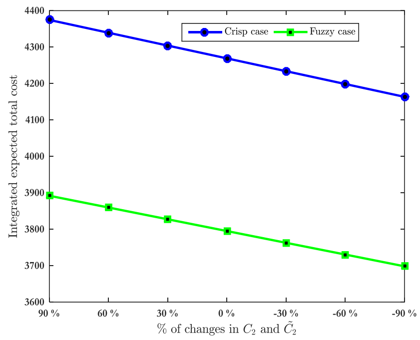
(e) Sensitivity analysis for M and $\tilde{M}$.



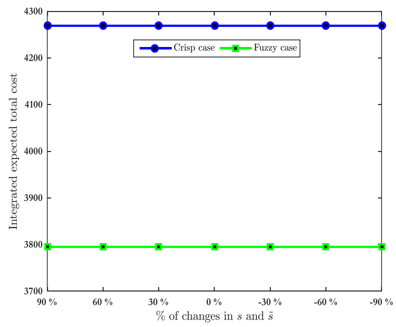
(f) Sensitivity analysis for n and $\tilde{n}$.



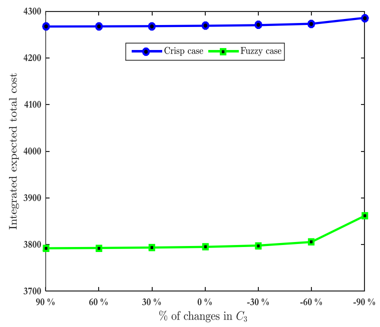
(g) Sensitivity analysis for C_1 and $\tilde{C}_1$.



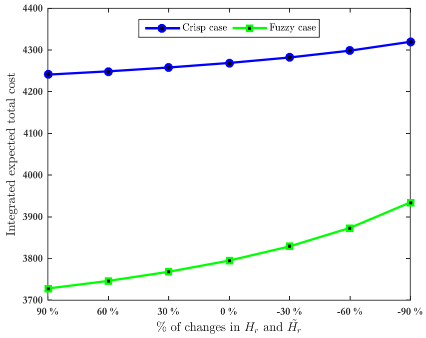
(h) Sensitivity analysis for C_2 and $\tilde{C}_2$.



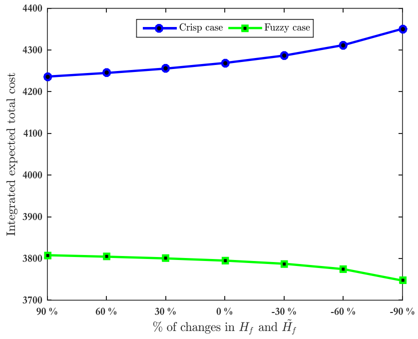
(i) Sensitivity analysis for s and $\tilde{s}$.



(j) Sensitivity analysis for C_3.



(k) Sensitivity analysis for H_r and $\tilde{H}_r$.



(l) Sensitivity analysis for H_f and $\tilde{H}_f$.

Figure 4(a)-(l): Parameters variation on the integrated expected total cost.

7. Comparative study

In Table 4, the projected model is compared with those of shortage level, optimal production quantity, and unit quantity's integrated expected total cost. As the measurement deviation of enactment methods among the cases declines, the integrated expected total cost also decreases. It is evident that the model with fuzzy optimal production lot sizes, shortage levels, and integrated expected total costs is more advanced in terms of consumption than the model with crisp parameters. All enactment measures of the model are significantly influenced by the hypothesis regarding fuzzy minimum integrated total costs. Additionally, the fuzzy inventory case is related to a particular case in the preceding model. The fuzzy inventory model with Kuhn-Tucker conditions and the graded mean integration method has a vital influence on the coordination enactment processes, and this fuzzy inventory model can forecast the enactment processes powerfully.

The proposed model followed procedures to find the minimum integrated expected total cost in fuzzy situation. We compare the proposed model with the previous model and the outcomes are tabulated. Hence a proficient outcome for the proposed fuzzy model is achieved.

Table 4 shows the result of the Priyan et. al. [22] inventory model in a fuzzy environment. They used production rate as a fuzzy number and got the shortage level, optimal production lot size, unit quantity's integrated expected total cost, and integrated expected total cost, which values 24.3 units, 231.1 units, $18.5/unit, and $4270.0, respectively. In our proposed fuzzified inventory model, we get the shortage level, optimal production lot size, unit quantity's integrated expected total cost, and integrated expected total cost, which values 22.4 units, 211.0 units, $18.0/unit, and $3795.0, respectively. Based on two inventory models, it is observed that the integrated expected total cost varies significantly.

Comparisons	Shortage level	Optimal production quantity	Unit quantity's integrated expected total cost	Integrated expected total cost
Crisp inventory model	24.1	229.8	18.6	4269.0
Priyan et.al. [22] model production rate considered as fuzzy number	24.3	231.1	18.5	4270.0
This proposed fuzzy inventory model	22.4	211.0	18.0	3795.0
Savings (%) with Priyan et.al.[22] model	8.0	8.7	2.8	11.1

Table 4: Summary of the comparisons.

In this way, our fuzzy integrated inventory model provides the business with the ability to deal with indeterminate inventory cost constraints. It is perceived that uncertain cost parameters give 8.0%, 8.7%, 2.8%, and 11.1% of the savings in shortage level, optimal production lot-size, integrated unit quantity's expected total cost, and the integrated expected total cost, respectively. Hence, the proposed fuzzified inventory model is the finest way to manipulate the inventory models.

8. Conclusions

Modern global developments in manufacturing skills have brought about an alteration in business. Rapid-fluctuating skills on the manufactured article have become essential for a likewise fast reaction from manufacturing businesses. Towards these ends, manufacturing industries need to choose suitable manufacturing policies, invention plans, industrial methods, effort section and instrument resources, machinery, inventory control, and tools. The assortment of conclusions is difficult, and resolution-making is also challenging at the present time. Decision architects in the manufacturing segment frequently face the difficult task of producing and managing inventory goods in an indeterminate atmosphere. This study discussed a two-level production-inventory system under fuzzy parameters and decision variables by implementing pentagonal fuzzy numbers. At this point, the manufacturer and the vendor effort under a combined station, in which an integrated result architect makes entirely conclusions towards, optimising the decision variables and minimising the integrated expected total cost for the whole supply chain.

An inventory model with scheduled fuzzy parameters and decision variables is offered in the proposed system. The pentagonal fuzzy inventory model was solved using Kuhn-Tucker conditions and framed the algorithm to examine the optimum result for decision variables and total cost for the proposed inventory model. According to the results, there was a variation in the cost between crisp and fuzzy models in response to fluctuations in the decision variables. Hence, the research exposes that in the model, the fuzzy model provided a commercial profit-making model for the minimum integrated expected total cost. For future investigation on this problem, we would consider multi-item with multi-objective of the system to deliberate the setup cost or ordering cost, for instance, as a decide variable and deliberate the special properties of decreasing ordering cost or setup cost changeability.

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