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# Group Fuzzy Languages and its Generalizations

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**Abstract**

In fuzzy language theory, every monoid is the syntactic monoid of some fuzzy language. By using this result the properties of fuzzy language can be studied by the algebraic properties of the syntactic monoids. There are so many methods for studying fuzzy languages. We adopt the above method to analyze different class of fuzzy languages and also characterize certain varieties of fuzzy languages. In this paper we give the variety description of group fuzzy languages and also provide the Eilenberg variety theorem for the class of group fuzzy languages. Moreover we described a fuzzy language whose syntactic monoid is isomorphic to the group $\mathbb{Z}_n$.

**Keywords**: syntactic congruence; group fuzzy language; pseudovariety.
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# 1 Introduction

The formal study of semigroups has started in the early 20th century. The most relevant and intensively studied classes of semigroups are finite semigroups, regular semigroups and inverse semigroups. The theory of finite semigroups has great importance in the theoretical computer science because of its natural link with finite automata; in the sense that semigroups can arise as syntactic semigroup of regular languages and as transition semigroups of finite automata. This concept is mainly useful to give characterization for regular languages by using the algebraic properties of their corresponding syntactic semigroups. Eilenberg [7] put forward the idea of theory of varieties of rational languages. He establishes a one to one correspondence between the varieties of formal languages and pseudo varieties of finite semigroups through his well known variety theorem.

The notion of fuzzy language was introduced by Lee and Zadeh [9]. It is a genealization of crisp language in the sense that the characteristic function has been replaced by membership function. Since the characteristic function $\chi_L$ of a language $L$ is a fuzzy language, every language is a fuzzy language.

The variety of fuzzy languages were introduced by T. Petkovic[11]. A class $F$ of fuzzy language over an alphabet $A$ is called a variety if it is closed under union, intersection, complements, multiplication by constants, quotients, inverse homomorphic images and $c$–cuts. He proved that there exists a one to one correspondence between variety of fuzzy languages and pseudo variety of monoids. This result is known as Eilenberg variety theorem for fuzzy languages. He also proved that there exists a mutually inverse lattice isomorphism between the lattices of all varieties of fuzzy languages and all varieties of crisp languages. Using this result and Eilenberg variety theorem, he has established a one-to-one correspondence between the lattice of all varieties of fuzzy languages and all pseudo varieties of finite monoids. One of the major result in fuzzy language theory has been considered as every monoid is the syntactic monoid of some fuzzy language[11]. Using this result we study certain classes of fuzzy languages ([2], [3], [4], [5], [6]). The aim of this paper is to provide a variety structure of group fuzzy languages. Also we give the Elienberg variety theorem for the class of group fuzzy languages. Moreover we described a fuzzy language whose syntactic monoid is isomorphic to the group $\mathbb{Z}_n$.

# 2 Preliminaries

Here we recall the basic definitions and notations that will be used in the sequel. All undefined terms are as in [10]. A non-empty set $S$ with an associative binary operation is called a semigroup. If there is an element $1 \in S$ with

$1s = s = s1$ for all $s \in S$, then $S$ is called a monoid (semigroup with identity).

**Definition 2.1** (cf. [7]). *A pseudovariety of monoids (resp. semigroups) is a class* $\mathbf{V}$ *of finite monoids (resp. semigroups) that is closed under finite direct product, morphic image and subobject. That is,*

*(i) If* $S, T \in \mathbf{V}$, *then* $S \times T \in \mathbf{V}$*;*

*(ii) If* $S \in \mathbf{V}$ *and* $\varphi : S \to S'$ *is a homomorphism, then* $\varphi(S) \in \mathbf{V}$*;*

*(iii) If* $S \in \mathbf{V}$ *and* $S'$ *is a submonoid (resp. subsemigroup) of* $S$, *then* $S' \in \mathbf{V}$.

Let $A$ be a non-empty finite set called an alphabet. Elements of $A$ are called letters. A word is a finite sequence of letters from $A$. The length of a word is the number of letters in it. A word of length zero is called the empty word, it is denoted by $1$. $A^+$ denote the set of all non-empty words. Then, $A^* = A^+ \cup \{1\}(A^+)$ together with the binary operation concatenation is called a free monoid(semigroup) on $A$.

A fuzzy language $\lambda$ in $A^*(A^+)$ is a fuzzy subset of $A^*(A^+)$. To each fuzzy language $\lambda$ we associate a congruence $P_\lambda$ called syntactic congruence, as follows:

For $u, v \in A^*(A^+)$, $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in A^*(A^+)$. The quotient monoid(semigroup), Syn$(\lambda) = A^*/P_\lambda(Syn\lambda = A^+/P_\lambda)$ is called the syntactic monoid (semigroup) of $\lambda$.

A fuzzy language $\lambda$ over an alphabet $A$ is recognizable by a monoid (semigroup) $S$ if there is a homomorphism $\phi : A^* \to S(\phi : A^+ \to S)$ and a fuzzy subset $\pi$ of $S$ such that $\lambda = \pi\phi^{-1}$, where $\pi\phi^{-1}(u) = \pi(\phi(u)$.

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet $A$, complement, union and intersection are defined respectively by

$$\begin{aligned} \overline{\lambda}(u) &= 1 - \lambda(u), \\ (\lambda_1 \vee \lambda_2)(u) &= \lambda_1(u) \vee \lambda_2(u), \\ (\lambda_1 \wedge \lambda_2)(u) &= \lambda_1(u) \wedge \lambda_2(u). \end{aligned}$$

Further left and right quotients are defined respectively by;

$$(\lambda_1^{-1}\lambda_2)(u) = \bigvee_{v \in A^*} (\lambda_2(vu) \wedge \lambda_1(v)),$$

$$(\lambda_2\lambda_1^{-1})(u) = \bigvee_{v \in A^*} (\lambda_2(uv) \wedge \lambda_1(v)).$$

Let $c \in [0, 1]$ be arbitrary. Then the fuzzy language $c\lambda$ defined by $(c\lambda)(u) = c \cdot \lambda(u)$ is called multiplication by a constant $c$.

Let $A, B$ be finite alphabets, $\phi : A^* \to B^*(\phi : A^+ \to B^+)$ be a homomorphism and $\psi$ a fuzzy language in $B^*(B^+)$, then the inverse image of $\psi$ (under $\phi$) is a fuzzy language $\psi\phi^{-1}$ over $A$ defined by $(\psi\phi^{-1})(u) = \psi(\phi u)$.

For a fuzzy language $\lambda$ by a $c$-cut, $c \in [0,1]$, we mean the crisp language $\lambda_c$ defined by $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$.

The following theorem gives a characterization for regular fuzzy languages.

**Theorem 2.1.** *[10] A fuzzy language $\lambda$ is regular if and only if $Im(\lambda)$ is finite and language $\lambda_c$ is regular for every $c \in [0,1]$.*

Unless otherwise specified all the fuzzy languages considered here are regular.

**Definition 2.2.** *A family $\mathscr{F} = \mathscr{F}(A)$ of regular fuzzy languages is a variety of fuzzy languages in $A^*(A^+)$ if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.*

**Theorem 2.2.** *Let $\lambda$ be a fuzzy language over an alphabet $A$. Then we have the following:*

*(i) Syn($\lambda$) recognizes $\lambda$;*

*(ii) If $M$ is any other monoid (resp. semigroup) recognizing $\lambda$, then Syn($\lambda$) is a homomorphic image of $M$.*

**Proof. (i)** Clearly $\eta_\lambda : A^* \to Syn(\lambda)$(resp. $A^+ \to Syn(\lambda)$) is an onto homomorphism. Define $\pi : Syn(\lambda) \to [0,1]$ by $\pi(\eta_\lambda(u)) = \lambda(u)$. Then $\lambda = \pi\eta_\lambda^{-1}$ and so Syn($\lambda$) recognizes $\lambda$.

**(ii)** Since $M$ recognizes $\lambda$, there is an onto homomorphism $\varphi : A^* \to M(A^+ \to M)$ and a function $\pi' : M \to [0,1]$ such that $\lambda = \pi'\varphi^{-1}$. Define $\psi : M \to Syn(\lambda)$ by $\psi(\varphi(u)) = \eta_\lambda(u)$. Then $\psi$ is an onto homomorphism, since $\eta_\lambda$ is an onto homomorphism. Thus Syn($\lambda$) is a homomorphic image of $M$. □

**Theorem 2.3.** *If the fuzzy languages $\lambda_1$ and $\lambda_2$ are recognizable by monoids (semigroups), then $\lambda_1 \vee \lambda_2$, $\lambda_1 \wedge \lambda_2$ are recognizable by monoids (semigroups).*

**Proof.** Let $M_1, M_2$ be monoids (semigroups), $\varphi_i : A^* \to M_i$ $(A^+ \to M_i), i = 1,2$ be onto homomorphisms and $\pi_i : M_i \to [0,1], i = 1,2$ be mappings such that $\lambda_i = \pi_i\varphi_i^{-1},\ i = 1,2$. Define $\varphi : A^* \to M_1 \times M_2$ $(A^+ \to M_1 \times M_2)$ by

$$\varphi(u) = (\varphi_1(u), \varphi_2(u))$$

and $\pi : M_1 \times M_2 \to [0,1]$ by

$$\pi(\varphi_1(u), \varphi_2(u)) = \pi_1(\varphi_1(u)) \vee \pi_2(\varphi_2(u)).$$

Then $\varphi$ and $\pi$ are well defined mappings and $\varphi$ is a homomorphism.

For all $u \in A^*$ (resp. $A^+$), we have

$$\begin{aligned}\pi\varphi^{-1}(u) = \pi(\varphi(u)) &= \pi(\varphi_1(u), \varphi_2(u)) \\ &= \pi_1(\varphi_1(u)) \vee \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \vee \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \vee \lambda_2(u) \\ &= (\lambda_1 \vee \lambda_2)(u).\end{aligned}$$

Thus, $\lambda_1 \vee \lambda_2 = \pi\varphi^{-1}$, and hence $\lambda_1 \vee \lambda_2$ is recognized by $M_1 \times M_2$. Now define $\pi' : M_1 \times M_2 \to [0, 1]$ by

$$\pi'(\varphi_1(u), \varphi_2(u)) = \pi_1(\varphi_1(u)) \wedge \pi_2(\varphi_2(u)).$$

Then $\pi'$ is a well defined mapping. For all $u \in A^*$ (resp. $A^+$), we have

$$\begin{aligned}\pi'\varphi^{-1}(u) = \pi'(\varphi(u)) &= \pi'(\varphi_1(u), \varphi_2(u)) \\ &= \pi_1(\varphi_1(u)) \wedge \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \wedge \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \wedge \lambda_2(u) \\ &= (\lambda_1 \wedge \lambda_2)(u).\end{aligned}$$

Thus $\lambda_1 \wedge \lambda_2 = \pi'\varphi^{-1}$ and hence $\lambda_1 \wedge \lambda_2$ is recognized by $M_1 \times M_2$. □

**Theorem 2.4.** *Let $\lambda_1, \lambda_2$ be fuzzy languages over over an alphabet $A$. If a monoid $M$ recognizes $\lambda_1$, then $(\lambda_2^{-1}\lambda_1)$ and $(\lambda_1\lambda_2^{-1})$ are recognized by $M$.*

**Proof.** Since $M$ recognizes $\lambda$, there exists an onto homomorphism $\varphi : A^* \to M$ and a function $\pi : M \to [0, 1]$ such that $\lambda_1(u) = \pi(\varphi(u))$ for all $u \in A^*$. Define $\pi_1 : M \to [0, 1]$ by

$$\pi_1(m) = \bigvee_{v \in A^*} (\pi(\varphi(vu)) \wedge \lambda_2(v)),$$

where $m = \varphi(u)$. Then $\pi_1(\varphi(u)) = \bigvee_{v \in A^*} (\lambda_1(vu)) \wedge \lambda_2(v)) = \lambda_2^{-1}\lambda_1(u)$. So $M$ recognizes $\lambda_2^{-1}\lambda_1$. Similarly $\lambda_1\lambda_2^{-1}$ is recognized by $M$. □

**Lemma 2.1** (cf. [11], Lemma 2)**.** *Let $A$ and $B$ be alphabets, $\lambda, \lambda_1, \lambda_2$ be fuzzy languages over $A$, $\tau$ be a fuzzy language over $B$, $\varphi : A^* \to B^*$ be a homomorphism and $c \in [0, 1]$ be a constant. Then*

*(i)* $P_{\overline{\lambda}} = P_{c\lambda} = P_\lambda$, $P_{\lambda_1 \vee \lambda_2}, P_{\lambda_1 \wedge \lambda_2} \supseteq P_{\lambda_1} \cap P_{\lambda_2}$;

*(ii)* $P_{\lambda_1^{-1}\lambda_2}, P_{\lambda_2\lambda_1^{-1}} \supseteq P_{\lambda_2}$;

*(iii)* $\varphi \circ P_\tau \circ \varphi^{-1} \subseteq P_\tau \varphi^{-1}$, *where* $\varphi \circ P_\tau \circ \varphi^{-1}$ *is a congruence on* $A^*$ *defined by* $(u, v) \in \varphi \circ P_\tau \circ \varphi^{-1} \Leftrightarrow (u\varphi, v\varphi) \in P_\tau$.

**Corolary 2.1.** *Let* $\lambda$ *be a fuzzy language over* $A$. *Then*

*(i)* $Syn(\overline{\lambda}) = Syn(\lambda)$ *and*

*(ii)* $Syn(c\lambda) = Syn(\lambda)$ *for all* $c \in [0, 1]$.

**Proof. (i)** By Lemma 2.1(i), we have $P_{\overline{\lambda}} = P_\lambda$. So $Syn(\overline{\lambda}) = Syn(\lambda)$.
**(ii)** Again by Lemma 2.1(i), $P_\lambda = P_{c\lambda}$ and hence $Syn(\lambda) = Syn(c\lambda)$ for all $c \in [0, 1]$. □

For a variety of fuzzy languages $\mathscr{F}$, let $\mathscr{F}^s$ be the family of finite monoids defined by $\mathscr{F}^s = \{\text{Syn}(\lambda) : \lambda \in \mathscr{F}(A), \text{for some} A\}$. For a variety of finite monoids $\mathscr{S}$, let $\mathscr{S}^f = \mathscr{S}^f(A)$ be the family of fuzzy languages defined by $\mathscr{S}^f(A) = \{\lambda$ is a fuzzy language over $A$ : Syn$(\lambda) \in \mathscr{S}\}$.

**Theorem 2.5.** *(cf.[11],Theorem 7) The mappings* $\mathscr{F} \to \mathscr{F}^s$ *and* $\mathscr{S} \to \mathscr{S}^f$ *are mutually inverse lattice isomorphisms between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

# 3 Group Fuzzy Languages

A fuzzy language $\lambda \in A^*$ is called group fuzzy language if its syntactic monoid (Syn($\lambda$)) is a group. Also if it is regular, then we call as regular group fuzzy language.

**Example 3.1** (cf. [8])**.** *Let* $G$ *be a finite group and* $\varphi : A^* \to G$ *be an onto homomorphism. If* $L = \varphi^{-1}(e)$, *then* $\chi_L$ *is a group fuzzy language.*

The class of regular group fuzzy languages in $A^*$ is denoted by **GFL**. By the above example **GFL** $\neq \phi$.
The following lemma shows that **GFL** is closed under the boolean operations.

**Lemma 3.1.** *Let* $\lambda, \lambda_1, \lambda_2 \in$ ***GFL***. *Then* $\overline{\lambda} \in$ ***GFL***, $\lambda_1 \vee \lambda_2 \in$ ***GFL*** *and* $\lambda_1 \wedge \lambda_2 \in$ ***GFL***.

**Proof.** Since Syn($\lambda$)= Syn($\overline{\lambda}$) (Corollary 2.1 (i)) and Syn($\lambda$) is a group, it follows that $\overline{\lambda} \in GFL$. Since Syn($\lambda_1$), Syn($\lambda_2$) are groups and $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are recognized by Syn($\lambda_1$) $\times Syn(\lambda_2)$ (Theorem 2.3), we see that $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are in **GFL**. □

Next result shows that **GFL** is closed under the inverse homomorphic images.

**Lemma 3.2.** *Let $A$ be finite alphabet, $\lambda$ be a fuzzy language on $A^*$ and $\lambda \in$ **GFL**. If $\varphi : X^* \to A^*$ is a homomorphism, then $\lambda\varphi^{-1} \in$ **GFL**, where $\lambda\varphi^{-1}(u) = \lambda(\varphi(u))$ for all $u \in X^*$.*

**Proof.** Since Syn($\lambda$) recognizes $\lambda$, there is a homomorphism $\psi : A^* \to Syn(\lambda)$ and a mapping $\eta : Syn(\lambda) \to [0,1]$ such that $\lambda = \eta\psi^{-1}$, where $\eta\psi^{-1}(u) = \eta(\psi(u))$. So $\psi \circ \varphi$ is a homomorphism from $X^*$ into the group Syn($\lambda$) and

$$\begin{aligned} \eta((\psi \circ \varphi)^{-1})(u) &= \eta((\psi \circ \varphi)(u)) \\ &= \eta(\psi(\varphi(u))) \\ &= \eta\psi^{-1}(\varphi(u)) \\ &= \lambda(\varphi(u)). \end{aligned}$$

So $\lambda\varphi^{-1}$ is recognized by Syn($\lambda$). Since Syn($\lambda\varphi^{-1}$) is a homomorphic image of Syn($\lambda$), we have $\lambda\varphi^{-1} \in$ **GFL**. □

**Lemma 3.3.** *Let $\lambda_1, \lambda_2 \in$ **GFL**. Then $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are in **GFL**.*

**Proof.** By Theorems 2.2 and 2.4, $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are recognized by Syn($\lambda_1$). Since Syn($\lambda_1$) is a group and Syn($\lambda_2^{-1}\lambda_1$), Syn($\lambda_1\lambda_2^{-1}$) are homomorphic images of Syn($\lambda_1$), we have $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are in **GFL**. Hence **GFL** is closed under the quotients. □

**Lemma 3.4.** *Let $\lambda \in$ **GFL** and $c \in [0,1]$. Then $c\lambda \in$ **GFL**.*

**Proof.** By Corollary 2.1(i), we have Syn($c\lambda$) = Syn($\lambda$). Since Syn($\lambda$) is a group, we see that $c\lambda \in$ **GFL**. □

Next result shows that **GFL** is closed in $c$-cuts.

**Lemma 3.5.** *Let $\lambda \in$ **GFL** and $c \in [0,1]$. Then the syntactic monoid of the language $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$ is a group.*

**Proof.** Since $G = Syn(\lambda)$. Then there is an onto homomorphism $\varphi : A^* \to G$ and a function $\pi : G \to [0,1]$ such that $\lambda = \pi\varphi^{-1}$. Let $P = \{m \in G : \pi(m) \geq c\}$. Then

$$\begin{aligned} \varphi^{-1}(P) &= \{u \in A^* : \varphi(u) \in P\} \\ &= \{u \in A^* : \pi(\varphi(u)) \geq c\} \\ &= \{u \in A^* : \pi\varphi^{-1}(u) \geq c\} \\ &= \{u \in A^* : \lambda(u) \geq c\} \\ &= \lambda_c. \end{aligned}$$

So $G$ recognizes $\lambda_c$. Since the syntactic monoid of $\lambda_c$ ($M(\lambda_c)$) is a homomorphic image of $G$, $M(\lambda_c)$ is a group. □

**Theorem 3.1.** *(i) **GFL** is a variety of fuzzy languages;*

*(ii) There exists a one-one correspondence between **GFL** and the pseudovariety **G** of finite groups.*

**Proof. (i)** Follows from Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5.
**(ii)** Let $\lambda \in$ **GFL**. Then by definition $Syn(\lambda) \in$ **G**.

Conversely, assume that $G \in$ **G**. Since every monoid is a syntactic monoid of some fuzzy language (cf. [11], Theorem 4), there is a fuzzy language $\lambda$ such that $Syn(\lambda) = G$. So $\lambda \in$ **GFL**. □

Next theorem gives an example of a fuzzy language whose syntactic monoid is a group.

**Theorem 3.2.** *Let $t \geq 1$ and $n > 0$. For $u \in A^*$, define $\lambda_n : A^* \to [0,1]$ by*

$$\lambda_n(u) = \frac{k}{k+t} \text{ if } |u| \equiv k (\mathrm{mod}\, n).$$

*Then Syn($\lambda_n$) is isomorphic to the group $\mathbb{Z}_n$ (under addition modulo $n$).*

**Proof.** Let $u, v \in A^*$ and let $uP_{\lambda_n}v$. Then $\lambda_n(xuy) = \lambda_n(xvy)$ for all $u, v \in A^*$. If $|xuy| \equiv k_1(\mathrm{mod}\, n)$ and $|xvy| \equiv k_2(\mathrm{mod}\, n)$, then by the definition of $\lambda_n$, we have $\dfrac{k_1}{k_1+t} = \dfrac{k_2}{k_2+t}$. So $k_1(k_2+t) = k_2(k_1+t)$. That is, $(k_1-k_2)t = 0$. Since $t \geq 1$, we have $k_1 = k_2$ and so $|xuy| \equiv |xvy|(\mathrm{mod}\, n)$. Since $|xuy| = |x|+|u+|y|$ and $|xvy| = |x|+|v|+|y|$, we see that $||u|-|v|| \equiv 0(\mathrm{mod}\, n)$. Thus $[u]_{\lambda_n} = \{v \in A^* \mid ||u|-|v|| \equiv 0(\mathrm{mod}\, n)\}$. If $w_0 = \epsilon$ and $w_i$ is a word of length $i$, then

$$[w_0] = \{\epsilon\} \bigcup \{u \in A^+ : |u| = kn,\ k = 1, 2, \ldots\}$$

and for $i = 1, 2, \ldots, n-1$

$$[w_i] = \{u \in A^+ : |u| = i + kn,\ k = 1, 2, \ldots\}.$$

So $Syn(\lambda_n) = \{[w_0], [w_1], \ldots, [w_{n-1}]\}$. Clearly Syn($\lambda_n$) is a monoid. Since $[w_0][w_i] = [w_i] = [w_i][w_0]$ for all $i$, it follows that $[w_0]$ is the identity element in Syn($\lambda_n$). If $i + j \equiv 0(\mathrm{mod}\, n)$, then $[w_i][w_j] = [w_iw_j] = [w_0] = [w_j][w_i]$. So $[w_j]$ is the inverse of $[w_i]$. Thus Syn($\lambda_n$) is a group. Define $\varphi : Syn(\lambda_n) \to \mathbb{Z}_n$ by $\varphi([w_i]) = |w_i|$ for all $i$. Then $\varphi$ is an isomorphism. □

# 4 Conclusion

In fuzzy language theory, every monoid is the syntactic monoid of some fuzzy language. With this concept we study different types of fuzzy languages with the algebraic properties of its syntactic monoids. In this paper we give the variety structure of group fuzzy language and also provide the Eleinberg variety theorem for group fuzzy language. We can extend this work into a distributive lattice $L$, since $L$-fuzzy languages are generalization of fuzzy languages[1].


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